Precalculus

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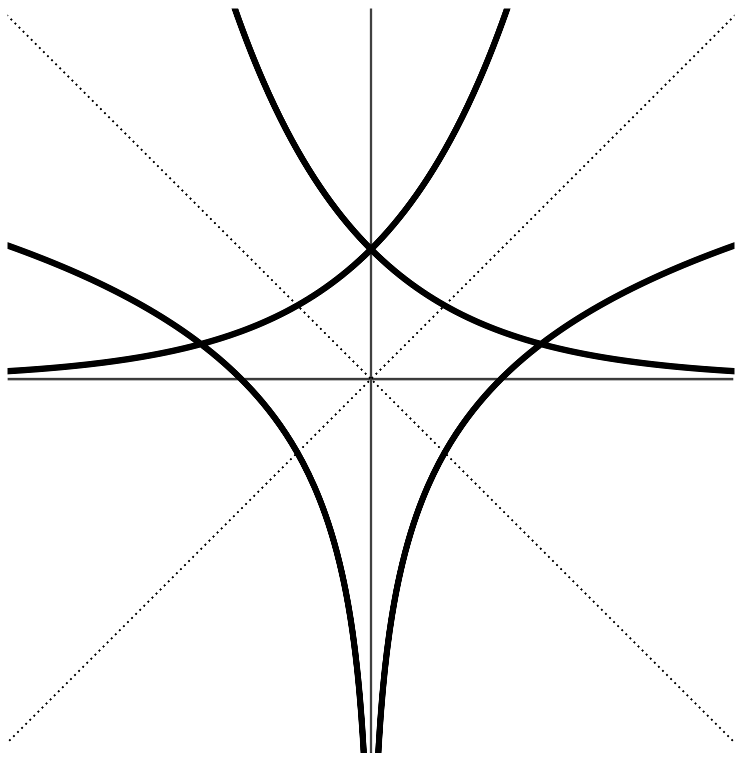


Edition $\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i$

August 25, 2017

"I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery."

—René Descartes, *La Géométrie*, 1637

Contents

1 Algebraic Aerobics

Section 1.1 Remember This?

•Linear Equations• Let's begin by considering the very familiar family of equations having the form

$$y = mx + b \tag{1.1}$$

Think About It 1.1.1 Write down at least five things you know about equations of the form $y = mx + b$.

While there are four *letters* in eq. (1.1), we typically refer to only two of these letters as **variables**. We refer to the other two as **parameters**.

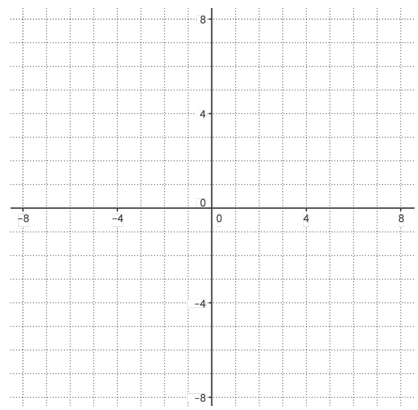
Think About It 1.1.2 Which of the letters in eq. (1.1) are variables and which are parameters? Which give equations of this form their name? Explain, in your own words, the different role that the variables and parameters have in eq. (1.1).

Perhaps you noted in your response to TAI 1.1.1 that the graph of any equation which can be written in the form $y = mx + b$ is a straight line. Another form in which equations of straight lines can be written is

$$y - y_o = m(x - x_o) \tag{1.2}$$

Think About It 1.1.3 What are the parameters in eq. (1.2)? What name do these suggest for this form of a linear equation?

Think About It 1.1.4 Write down the value of each parameter in the equation $y-6=-\frac{2}{5}(x+7)$, thinking carefully about the signs. Then, *without doing any algebraic manipulation*, graph the equation.



Think About It 1.1.5 Is it possible to change the value of one or more of the given parameters in TAI 1.1.4 and still have an equation which has the same graph? If so, give an example. If not, explain why it is impossible.

Think About It 1.1.6 What's the definition of the slope of a line? How is this related to eq. (1.2)?

Think About It 1.1.7 Equation (1.1) is actually just a **special case** of eq. (1.2). Which special case is it?

•Logarithms are Exponents •

Think About It 1.1.8 If m is the exponent you would put on 2 to get 24, and n is the exponent you would put on 2 to get $\frac{1}{3}$, what is $m+n$?

(If, after experimenting with the question above for a while, you feel stuck, replace 24 with larger but "nicer" number and replace $\frac{1}{3}$ with a smaller but "nicer" number. Answer your new question and then try the original one again.)

Exponent & Log Rules (assuming $a > 0$)

$$a^b \cdot a^c = \qquad\qquad \log_a MN =$$

$$\frac{a^b}{a^c} = \qquad\qquad \log_a \frac{M}{N} =$$

$$\left(a^b\right)^c = \qquad\qquad \log_a M^c =$$

Think About It 1.1.9 Let $a^b = M$ and let $a^c = N$, and try to show how to obtain each log rule from the corresponding exponent rule.

It turns out that a surprisingly useful base for logarithms is a transcendental number that pops up repeatedly in advanced mathematics and whose discovery is credited to the 17$^{\text{th}}$ century Swiss mathematician Jacob Bernoulli. The person who introduced the name by which we know this number today, e, was Leonhard Euler, born in Switzerland 2 years after Bernoulli died. Your calculator will give you as good an approximation to e as most people are ever likely to need. Because e is at least as important mathematically as its more widely known transcendental cousin π its worth knowing the first few digits.[1]

$$e \approx$$

Because e turns out to be so useful as a base, a logarithm with the base of e is given the special name **natural log** and a special notation: instead of writing $\boldsymbol{\log_e M}$ we typically write $\boldsymbol{\ln M}$.

Another logarithm which gets a special name is that with a base of 10. It is known as the **common logarithm** and the special notation is to omit specifying the base at all. That is, instead of writing $\boldsymbol{\log_{10} M}$ we typically write $\boldsymbol{\log M}$. (Just to add to the confusion, in some contexts, when log is written without a base specified it refers to the natural log. Try typing $\log 100$ into Wolfram Alpha and see what happens.)

Think About It 1.1.10 What is that value of each expression?

$\ln e^3$	$\ln(-e)$	$\log 0$
$\ln \frac{1}{e}$	$\log 0.01$	$\log 5 + \log 20$
$\ln \sqrt{e}$	$\log 1$	$\log 70 - \log 7$

Think About It 1.1.11 What two consecutive integers is the value of $\log 7831$ in between? How about $\log_2 40$? $\log_3 \frac{1}{10}$?

•Intercepts and Roots/Zeros• Having come this far in your mathematics education, you are undoubtedly very familiar with the concept of intercepts, the points on the graph of a function (or relation) which fall on one of the coordinate axes.

Think About It 1.1.12 Explain, in general, how to determine the intercepts of a function without actually graphing it if you are given the equation of the function.

[1] If you're wondering what a transcendental number is exactly, you should watch this Numberphile video (youtu.be/seUU2bZt-fgM)

Think About It 1.1.13 When y is a function of x, is it typically easier to find the x-intercepts or the y-intercepts or are they equally easy (or difficult) to find? Explain.

Definition 1.1.1 a is said to be a **root** (or a **zero**) of a function if and only if $f(a) = 0$.

Think About It 1.1.14 Say something about the connection between roots (or zeros) and intercepts.

•Fractions Signify Division• As you are undoubtedly aware, $\dfrac{48}{8}$ is just another way of writing $48 \div 8$. You also know that dividing by 2 is the same as multiplying by $\frac{1}{2}$ and *vice-versa*. Remembering these facts is of great help in simplifying compound fractions (as in Example 1.1.1 below).

Also, if you happen to be one who is tempted to "simplify" an expression like $\dfrac{40+2x}{2x}$ by crossing out the $2x$'s, perhaps you can avoid the temptation by thinking about the fact that when you do this, you are asserting that *division* undoes *addition*, which in turn implies that there is no difference between the expressions $\dfrac{40+2x}{2x}$ and $(40+2x) - 2x$. This, of course, is utter nonsense.

Example 1.1.1 Simplify: $\dfrac{\dfrac{2}{x+5} - \dfrac{2}{x}}{5x+25}$

Solution Rewriting the given expression in the form "primary numerator ÷ primary denominator" gives

$$\left(\frac{2}{x+5} - \frac{2}{x}\right) \div (5x+25)$$

To perform the subtraction in the first set of parentheses, we must first write the two fractions with the common denominator of $x(x+5)$:

$$\left(\frac{2}{x+5} \cdot \frac{x}{x} - \frac{2}{x} \cdot \frac{x+5}{x+5}\right) \div (5x+25)$$

$$\left(\frac{2x}{x(x+5)} - \frac{2x+10}{x(x+5)}\right) \div (5x+25)$$

Simplifying the numerator of the single fraction remaining and rewriting the division as multiplication (this is where it helps to remember that dividing by 2 is the same as multiplying by $\frac{1}{2}$) gives

$$\frac{-10}{x(x+5)} \cdot \frac{1}{5x+25}$$

Note that the denominator of the second fraction can be factored:

$$\frac{-10}{x(x+5)} \cdot \frac{1}{5(x+5)}$$

That factor of 5 in the denominator should call our attention to the fact that there is also a factor of 5 in the numerator. Showing this factor explicitly and writing the whole expression as a single fraction gives

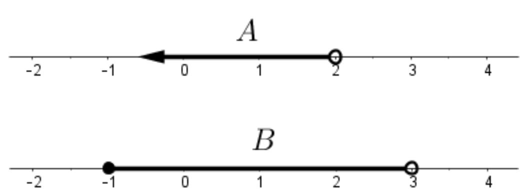
$$\frac{-2 \cdot 5}{5 \cdot x(x+5)^2}$$

Now that the numerator and denominator are both fully factored, we can take advantage of the fact that division undoes multiplication to write the fully simplified result as

$$\frac{-2}{x(x+5)^2}$$

Think About It 1.1.15 A quicker method of solution for Example 1.1.1 is to multiply the original fraction by 1 in the form of $\dfrac{x(x+5)}{x(x+5)}$. Show how this would work.

•Sets and Intervals• Consider the two sets of numbers represented by the shaded regions on the number lines below.



In **set-builder notation** we would designate Set A by writing $\{x \mid x < 2\}$, which is read "the set of all x such that x is less than 2." Set B would be designated as $\{x \mid -1 \leq x < 3\}$, "the set of all x such that x is greater than 1 and ≤ 3. (Any variable could be used in place of x.)

In **interval notation** we would specify Set A by writing $(-\infty, 2)$, and Set B by writing $[-1, 3)$. Can you surmise the rules for interval notation based on these two examples?

A few key set symbols

$\mathbb{N}$	$\mathbb{Z}$
$\mathbb{Q}$	$\mathbb{R}$
$\varnothing$	$\in$
$\cup$	$\cap$
$\subset$	$\subseteq$

Note: Some texts include 0 as a natural number and some do not. We will use the convention used by the IB, which is that 0 *is* a natural number. The IB uses the notation $\mathbb{Z}^+$ to represent the positive integers.

Think About It 1.1.16 Assuming that A and B are the sets illustrated above, how would you use set-builder notation and interval notation to specify $A \cup B$? $A \cap B$?

Think About It 1.1.17 Assuming that A and B are the sets illustrated above, find a simpler way of specifying each of the following: $A \cap \mathbb{N}$, $B \cap \mathbb{Z}$, $\mathbb{Z} \cup \mathbb{Q}$, $\mathbb{R} \cap \mathbb{Q}$

Problem Set 1.1

1. There are some straight lines whose equations cannot be written in either the form $y = mx + b$ or the form $y - y_o = m(x - x_o)$. Write a brief paragraph explaining what these lines have in common and why is it impossible to write their equations in either of these forms. Can you come up with a form in which makes it possible to write the equation of *any* straight line? What information is provided by the parameters of an equation in this form?

2. Substitute some numbers into the equations $a^m \cdot a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$ to help you write an explanation of why it makes sense for fractional and negative exponents to behave as they do.

3. Watch Vi Hart's 9-minute "How I Feel About Logarithms" video (bit.ly/VHLogs) and write a paragraph or two of explanation and/or reflection on ONE of the following statements from the video. **Include mention of how you think this statement in context is relevant to Hart's goal of helping the viewer to understand logarithms.**

- "Sometimes to make the harder things simple, you have to make simple things hard."
- "Exponentiation, powers . . . also fancy counting."
- "I think it's kind of weird that we keep around this root notation when fractional powers are so much more descriptive."
- "Can you feel the seven-sixthsness of 128 in relation to 64?"
- "This, my good friend, is the logarithmic scale."

In problems 4 to 9, write down the *slope-intercept* form of the equation of the line that has the given features.

4. has a slope of 3 and a y-intercept of -7

5. has a slope of -1 and a y-intercept of 0

6. passes through the points $(21, -3)$ and $(-5, -11)$

7. passes through the points $(45, -73)$ and $(45, 27)$

8. has an x-intercept of $-\frac{9}{5}$ and a y-intercept of $\frac{7}{3}$

9. has a y-intercept of 4 and is perpendicular to the line $2x - 5y = 79$

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10. Write two different equations in *point-slope form* for the line of slope 2 which contains the point $(-3, 7)$.

11. Write two different equations in *point-slope form* for the line of slope -3 which contains the point $\left(-8, -\frac{1}{5}\right)$.

12. Write an equation in point-slope form and an equation in slope-intercept form for the line that passes through the points $(-15, -7)$ and $(5, 22)$.

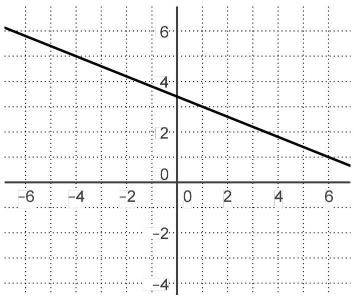
13. Write an equation in any form you like for the line of slope $\frac{5}{12}$ which contains the point $(18.76, -9.49)$.

14. Consider line ℓ which is perpendicular to the line $5x + 2y = 233$ and passes through $(17.8, -83.1)$.

a) Write an equation in any form you like for ℓ.
b) What is the y-coordinate of the point on ℓ whose x-coordinate is 27.8?
c) What is the x-coordinate of the point on ℓ whose y-coordinate is -85.1?

In problems 15 to 17, write an equation in point-slope form (*not slope-intercept form*!) for the line shown.

15.



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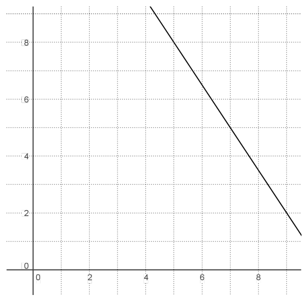

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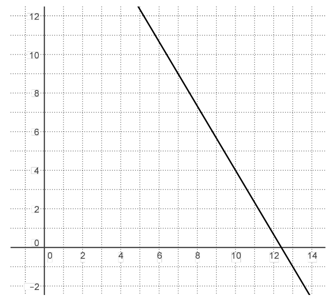

In problems 18 to 45, evaluate the expression without a calculator.

18. $\dfrac{4^0}{10^{-3}}$

19. $(-8)^{5/3}$

20. $\left(\dfrac{49}{64}\right)^{-1/2}$

21. $\left(-\dfrac{1}{1000}\right)^{2/3}$

22. $\left(\dfrac{25}{36}\right)^{-1/2}$

23. $\dfrac{2}{4^{-2}}$

24. $\left(-\dfrac{1}{27}\right)^{2/3}$

25. $\left(16^{1/2}\right)\left(9^{-2}\right)$

26. $\dfrac{5^{-2}}{(-3)^0}$

27. $\log_3 81$

28. $\ln 1$

29. $\log_4\left(\frac{1}{64}\right)$

30. $\log_{32} 2$

31. $\log_3(-9)$

32. $\log_2(-8)$

33. $\log(-\frac{1}{10})$

34. $\log 0.01$

35. $\log \sqrt[4]{10}$

36. $\log_5 \sqrt[3]{25}$

37. $\ln \sqrt[4]{e}$

38. $\log_{25} \sqrt[3]{5}$

39. $\log_\pi\left(\pi^{10}\right)$

40. $\log_5 1$

41. $\log_{0.25} 16$

42. $\log_8 4$

43. $\log_4 32$

44. $3^{\log_3 7}$

45. $2\log_6 3 + \log_6 4$

46. If m is the exponent you would put on 3 to get 54, and n is the exponent you would put on 3 to get $\frac{1}{6}$, what is $m+n$?

47. If m is the exponent you would put on 8 to get $\frac{10}{7}$, and n is the exponent you would put on 8 to get $\frac{7}{5}$, what is $m+n$?

48. If m is the exponent you would put on 3 to get $\frac{2}{3}$, and n is the exponent you would put on 3 to get 18, what is $m-n$?

In problems 49 to 55, find a and b if a and b are consecutive integers such that $a<b$ and

49. $a < \log_5 499 < b$

50. $a < \log_3 90 < b$

51. $a < \log 399 < b$

52. $a < \log 5 < b$

53. $a < \log_2\left(\dfrac{1}{10}\right) < b$

54. $a < \log_5\left(\dfrac{1}{100}\right) < b$

55. $a < \log_{0.5}\left(\dfrac{1}{3}\right) < b$

56. Find a value of A for which $\log_2 A$ will be

a) undefined

b) greater than 5 but less than 6

c) greater than -4 but less than -3

d) positive and less than $\frac{1}{2}$

57. Expand using log rules: $\log_2 \sqrt{8xy^3}$

58. Write the expression $2\log 3 + \log(x-3) - \log\left(x^2-9\right)$ as a single logarithm.

59. Find $A(x)$ if $\ln\left(2\sqrt{x^3}\right) + \ln 6 - 2\ln\left(x^{1/4}\right) = \ln A(x)$

60. If $\log P = 2\log 6x + \log x - \frac{1}{2}\log 16$, find P in simplest form.

In problems 61 to 64, solve the equation for x without using a calculator.

61. $4\ln 2 - 3\ln 4 = -\ln x$

62. $\log_3(x+17) - 2 = \log_3(2x)$

63. $\log_2(x+3) + \log_2(x+1) = 3$

64. $3^{2x} - 5\cdot 3^x = -6$

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65. Show that $\log_a N = \dfrac{\log_b N}{\log_b a}$.

(Hint: One possible approach is to begin by letting $y = \log_a N$ and then rewriting this expression in exponential form.)

66. $\log_2 3\cdot\log_3 4\cdot\log_4 5\cdot\cdots\cdot\log_{31} 32$

In problems 67 to 69, simplify the expression. Write your final answer with no negative exponents.

67. $\dfrac{\left(27x^{-9}y^3\right)^{2/3}}{\left(32x^5y^{-10}\right)^{-1/5}}$

68. $\left(\dfrac{16x^{-5}y^{1/4}}{xy^{-7/4}}\right)^{-3/2}$

69. $\dfrac{\left(4x^{3/2}y^{-1/2}\right)^{-1}}{\left(25x^{5/3}y^{-1/3}\right)^{3/2}}$

In problems 70 to 82, determine the x- and y-intercepts of the graph of each equation. If either does not exist, explain how you know. Give an exact answer in each case and when the exact answer is irrational, also state the two consecutive integers between which the exact value lies. (You do not actually need to draw the graph.)

70. $y = x^2 - 3x - 10$

71. $y = x^2 + 10x + 21$

72. $y = x^2 - 5x - 3$

73. $y = 2x^2 - 4x + 1$

74. $y = 5x^2 + 4x - 2$

75. $y = \sqrt{x-5}$

76. $y = 3 - \sqrt{x-4}$

77. $y = 3^x + 5$

78. $y = 2^x - 8$

79. $y = 2^x - 16$

80. $y = 3^x - 12$

81. $y = 2^x - 10$

82. $y = 100 - 5^x$

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83. Write out in words what you say when reading aloud the expression $A = \{x \mid x \leq 5\}$.

84. Write out in words what you say when reading aloud the expression $B = \{n \mid n \geq 10\}$.

85. If $A = \{x \mid -4 < x \leq 3\}$ and $B = \{x \mid x > 1\}$, find the following. Express your answers in set-builder notation and graph each result on a number line.

a) $A \cap B$
b) $A \cup B$

86. If $A = \{x \mid -1 \leq x < 5\}$ and $B = \{x \mid x < 2\}$, find the following. Express your answers in set-builder notation and graph each result on a number line.

a) $A \cup B$
b) $A \cap B$

87. If $A = \{x|x < -4\}$ and $B = \{x|-6 < x < 2\}$, find the following. Express your answers in set-builder notation and graph each result on a number line.

a) $A \cup B$
b) $A \cap B$

In problems 88 to 94, simplify each expression so that the result is a single fraction which doesn't itself contain fractions.

88. $\dfrac{1 - \dfrac{4}{x+1}}{\dfrac{x}{x+1} - \dfrac{3}{x}}$

89. $\dfrac{\dfrac{y}{y+1} - \dfrac{1}{2}}{\dfrac{1-y}{2y+1}}$

90. $\dfrac{\dfrac{3}{2a+3} - \dfrac{1}{a}}{2 - \dfrac{4}{2a+3}}$

91. $\dfrac{\dfrac{1}{x} - \dfrac{1}{4}}{x-4}$

92. $\dfrac{\dfrac{1}{x+3} - \dfrac{1}{4}}{x-1}$

93. $\dfrac{\dfrac{5}{x+h} - \dfrac{5}{x}}{h}$

94. $\dfrac{2x-5}{\dfrac{1}{2x} - \dfrac{1}{5}}$

Section 1.2 Sequences, Series, and Summation Notation

Exploration 1.1

Go to visualpatterns.org and pick a pattern to explore. Draw the next picture in the sequence. Try to come up with an equation that will generate the number of shapes in the n^{th} picture in the sequence. Check that your equation works when $n = 1$, when $n = 4$, and when $n = 43$.

When we specify a **set** of numbers, we do not require that the numbers be in any particular order as part of our definition of the set. A list of numbers in a particular order, on the other hand, is known as a **sequence**.

When we recognize a pattern in a sequence that we are working with, it is typically helpful to write a general expression for the n^{th} term in the sequence. Consider, for example, the sequence

$$\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots$$

If the pattern we see in this sequence is that the first term has a denominator of 2^1, while the second term has a denominator of 2^2, the third term has a denominator of 2^3, and so on, we might define the sequence by writing

$$a_n = \frac{5}{2^n}$$

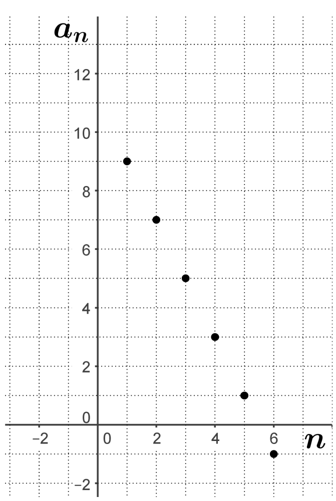
This expression—called an **explicit** definition of the sequence—enables us to plug in any value of n and calculate what the n^{th} term will be without knowing intermediate values. Another way you might have thought of the pattern in the sequence above was that each term is one-half of the previous term. To describe the pattern this way, we can write what's known as a **recursive** definition. A recursive definition is one which states the first term (or terms) of sequence and then provides a rule for getting each subsequent term from the one (or ones) before it. A recursive definition for the sequence we're discussing here would be

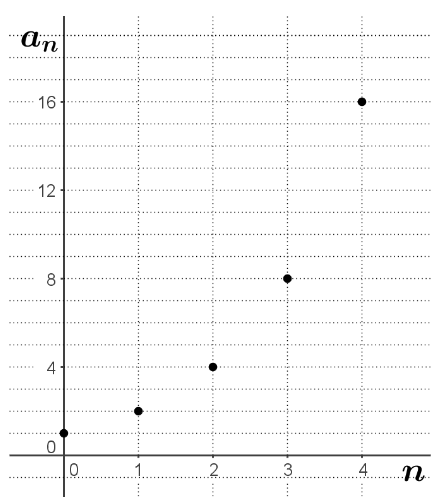
$$a_1 = \frac{5}{2}; \quad a_{n+1} = \frac{1}{2}a_n$$

Think About It 1.2.1 Find a_6 if $a_n = \dfrac{3n+1}{10n}$ and b_6 if $b_1 = -24$ and $b_{n+1} = \frac{1}{2}b_n + 4$.

Think About It 1.2.2 Explain why two statements are required in a recursive definition of a sequence while only one statement is necessary for an explicit definition.

Think About It 1.2.3 Write an explicit definition and a recursive definition for each of the sequences graphed below.





A sequence where you get to the next term by adding the same (negative or positive) amount at each step is called an **arithmetic sequence**. The amount you add to get from one term to the next in an arithmetic sequence is called the **common difference**. A sequence where you get to the next term by multiplying (or dividing, if you prefer) by the same amount at each step is called a **geometric sequence**. The amount you multiply by to get from one term to the next in a geometric sequence is called the **common ratio**.

When graphed as in TAI 1.2.3 the terms of an arithmetic sequence fall along a line, while the terms of a geometric sequence with a positive common ratio fall along an exponential curve. (Do you see why each of these is the case? What would you see if graphed the terms of a geometric sequence with a negative common ratio?) Note that when graphing the terms of a sequence you do not actually draw a line that goes through all of the dots. This is because there is no such thing, for example, as term $a_{\frac{1}{2}}$.

Think About It 1.2.4 Consider the sequence $-5, -1, 3, 7, \ldots$

Determine whether the sequence is arithmetic or geometric and state the common difference or the common ratio.

Write a recursive formula for this sequence.

Write an explicit formula for this sequence.

Identify the 5th term of the sequence, the 6th term of the sequence, and the 50th term of the sequence.

Think About It 1.2.5 Consider the sequence $400, 200, 100, 50, \ldots$

Determine whether the sequence is arithmetic or geometric and state the common difference or the common ratio.

Write a recursive formula for this sequence.

Write an explicit formula for this sequence.

Identify the 8th term of the sequence.

Think About It 1.2.6 Consider the general arithmetic sequence with a first term of a_1 and a difference between terms of d as well as the general geometric series with a first term of a_1 and a ratio between terms of r. Find expressions for a_2, a_3, and a_{10} in each case. Then find an explicit definition of a_n in each case.

Exploration 1.2

Write out an arithmetic sequence with 10 terms. Then find the following sums: $a_1 + a_{10}$, $a_2 + a_9$, $a_3 + a_8$, $a_4 + a_7$, and $a_6 + a_5$. What do you notice? Why is this? How might this observation help if you wanted to find the sum of an arithmetic sequence with 100 terms? What about a sequence with 101 terms?

•Series• The sum of the terms of a sequence is called a **series**. The most obvious way to write a series is to replace the commas in the sequence with plus signs. And if you want to indicate that there are a bunch of terms you're including but don't want to take the time to write, you can use an ellipsis ($\cdots$), like so:

$$\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\cdots+\frac{5}{65536}$$

The notation S_n is typically used to represent the sum of the first n terms in a sequence, so, for example, in this series S_2 would be $\frac{15}{4}$.

Think About It 1.2.7 Fill in the missing column entries to continue the patterns.

i	1	2	3	4	$\cdots$	10	$\cdots$	25	$\cdots$	n
a_i	2	5	8		$\cdots$	29	$\cdots$		$\cdots$	
S_i	2	7	15		$\cdots$	155	$\cdots$		$\cdots$	

Think About It 1.2.8 Verify that a geometric series, S_n, with a first term of a_1 and a ratio between subsequent terms of r can be written as

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

Now multiply both sides of this equation by r. If you subtract the resulting equation from the original equation and solve for S_n you'll discover a fairly simple formula for determining the sum of a geometric series.

Exploration 1.3

Consider a geometric series with a first term of 2. Use a calculator or computer to help you find the values of S_1 through S_{10} if the common ratio is $\frac{3}{2}$ and then again if the common ratio is $\frac{1}{5}$. What seems particularly noteworthy about the difference between these two cases? How can you determine which way a particular geometric series is going to behave? Use your observations along with the formula you discovered in TAI 1.2.8 to find the sum of *all* of the terms in an infinite geometric series that behaves like the case you studied in which common ratio was $\frac{1}{5}$.

•Summation Notation• Leonhard Euler (pronounced "Oy-ler"), a phenomenally prolific and impressive Swiss mathematician whose life spanned most of the 18^{th} century, whom you encountered briefly in the discussion of e in the previous section, and whose name and century are worth remembering, made up a shorthand for writing a series which used the capital Greek sigma. Can you figure out how the notation works by perusing the following examples?

$$\sum_{i=1}^{16} \frac{5}{2^i} \quad \text{represents} \quad \frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\cdots+\frac{5}{65536} \tag{1.3}$$

$$\sum_{j=2}^{6} \frac{5}{2^j} \quad \text{represents} \quad \frac{5}{4}+\frac{5}{8}+\frac{5}{16}+\frac{5}{32}+\frac{5}{64} \tag{1.4}$$

$$\sum_{t=100}^{102} 6 \quad \text{represents} \quad 6+6+6 \tag{1.5}$$

$$\sum_{k=-3}^{1} \frac{5}{2^k} \quad \text{represents} \quad 40+20+10+5+\frac{5}{2} \tag{1.6}$$

$$\sum_{n=-2}^{4} 10^n \quad \text{represents} \quad 0.01+0.1+1+10+100+1000+10,000 \tag{1.7}$$

$$\sum_{m=3}^{\infty} 4m+1 \quad \text{represents} \quad 13+17+21+25+\cdots \tag{1.8}$$

$$\sum_{x=0}^{4} mx+b \quad \text{represents} \quad b+(m+b)+(2m+b)+(3m+b)+(4m+b) \tag{1.9}$$

In the sum $\sum_{i=1}^{16} \frac{5}{2^i}$, we say that the **index of summation** is i, the **lower limit** is 1, the **upper limit** is 16 and the **general element** is $\frac{5}{2^i}$. When we write out the individual terms rather than using summation notation, we are writing the series in **expanded form**.

Think About It 1.2.9 Give the expanded form of the sequence which, when written in summation notation, has the general element $(-1)^t(2t)$, a lower limit of 3, and an upper limit of 7.

Think About It 1.2.10 Find two different ways of writing the summation notation for an arithmetic sequence with a first term of 5 and a common difference of 2. Can you do the same for the general arithmetic series with a first term of a and a common difference of d? How about the general geometric series with a first term of a and a common difference of r?

Problem Set 1.2

1. Where was Leonhard Euler from, when did he live, and how do you pronounce his last name? What's something else that he did besides introducing sigma notation for series?

2. Write a paragraph comparing and contrasting the formula for generating the terms of an arithmetic sequence with the formula for generating the terms of a geometric sequence. Explain how the similarities and differences reflect the essence of these two types of sequences.

3. Consider the sequence defined explicitly by $a_n = 5 - 2n$ for $n \geq 1$. Write out the first five terms of the sequence and give a recursive definition for the sequence.

4. Consider the arithmetic sequence $-11, 2, \ldots$

   a) Write a recursive formula for the sequence.
   b) Write an explicit formula for the sequence.
   c) What is the 50th term of the sequence?

5. Consider the arithmetic sequence where the first term is 7 and the fourth term is 11.

   a) Find the common difference.
   b) Write a recursive formula for the sequence.
   c) Write an explicit formula for the sequence.
   d) Identify the 20th term of the sequence.

6. Consider the arithmetic sequence where the first term is 8 and the fourth term is 12.

a) Find the common difference.
b) Write a recursive formula for the sequence.
c) Write an explicit formula for the sequence.
d) Identify the 19th and 20th terms of the sequence.

7. Consider the arithmetic sequence where the fourth term is 12 and the seventh term is 8.

a) Find the common difference.
b) Find the first term of the sequence.
c) Write a recursive formula for the sequence.
d) Write an explicit formula for the sequence.
e) Identify the 20th term of the sequence.

8. Consider the arithmetic sequence where the 17th term is 12 and the 22nd term is 8.

a) Find the common difference.
b) Find the first term of the sequence.
c) Write a recursive formula for the sequence.
d) Write an explicit formula for the sequence.
e) Identify the 20th term of the sequence.

9. Write an explicit definition and a recursive definition for the geometric sequence with a first term of 5 and a second term of 4.

10. Write an explicit definition and a recursive definition for the geometric sequence with a first term of 7 and a second term of 3.

11. Consider the geometric sequence where the fourth term is 12 and the seventh term is 8.

a) Find the common ratio.
b) Find the first term of the sequence.
c) Write a recursive formula for the sequence.
d) Write an explicit formula for the sequence.
e) Find the approximation of the 9th term of the sequence, correct to three places after the decimal.

12. Find a_{101} for the arithmetic sequence with $a_1 = 20$ and $a_5 = 8$.

13. Find a_{501} for the arithmetic sequence with $a_1 = 12$ and $a_4 = -3$.

14. Find the sum of the first 50 odd numbers.

In problems 15 to 22, express the sum using sigma notation.

15. $\frac{7}{8} + \frac{7}{9} + \frac{7}{10} + \frac{7}{11} + \frac{7}{12}$

16. $\frac{2}{9} + \frac{3}{16} + \frac{4}{25} + \frac{5}{36}$

17. $\frac{\sqrt{5}}{30} + \frac{\sqrt{6}}{40} + \frac{\sqrt{7}}{50} + \frac{\sqrt{8}}{60}$

18. $\frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8}$

19. $\sqrt{7} + \sqrt[3]{8} + \sqrt[4]{9} + \sqrt[5]{10} + \sqrt[6]{11}$

20. $0.05 + 0.5 + 5 + 50$

21. $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$

22. $10 + 12 + 14 + 16 + 18 + \cdots$

In problems 23 to 42, find the indicated sum. (In those cases where i appears, but is not the index of summation, it is representing $\sqrt{-1}$.)

23. $\displaystyle\sum_{j=2}^{4}(10j+3)$

24. $\displaystyle\sum_{j=-1}^{1}(10j+3)$

25. $\displaystyle\sum_{i=0}^{3}(3i-5)$

26. $\displaystyle\sum_{i=2}^{5}(3i-5)$

27. $\displaystyle\sum_{i=4}^{14}5$

28. $\displaystyle\sum_{i=9}^{90}10$

29. $\displaystyle\sum_{k=-1}^{3}k^2$

30. $\displaystyle\left(\sum_{k=-1}^{3}k\right)^2$

31. $\displaystyle\sum_{k=-1}^{3}2^k$

32. $\displaystyle\sum_{k=1}^{-2}\frac{k-1}{3^k}$

33. $\displaystyle\sum_{k=-2}^{0}\left(\frac{k}{2^k}+1\right)$

34. $\displaystyle\sum_{i=2}^{50}(-1)^i$

35. $\displaystyle\sum_{i=3}^{50}(-1)^i$

36. $\displaystyle\sum_{n=1}^{4}i^n$

37. $\displaystyle\sum_{n=2}^{6}i^n$

38. $\displaystyle\sum_{n=1}^{1000}i^n$

39. $\displaystyle\sum_{n=1}^{567}i^n$

40. $\displaystyle\sum_{n=0}^{400}i^n$

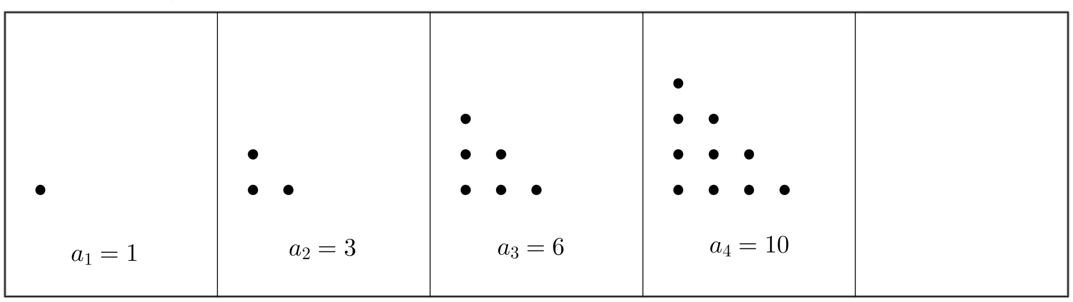
41. $\displaystyle\sum_{n=0}^{49}i^n$

42. $\displaystyle\sum_{n=3}^{101}i^n$

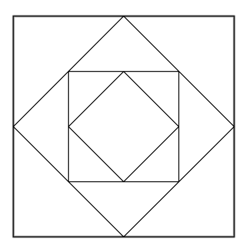
---

43. Find the value of a if $\displaystyle\sum_{i=3}^{7}2=\sum_{k=0}^{a}2$

44. Fill in the empty block to continue the pattern. What will the values of a_{10}, a_{100}, and a_n be?



45. Inside a 12-by-12 square we construct a smaller square whose vertices are the midpoints of the sides of the original square. Then we repeat the process over and over. The first 4 squares are shown in the diagram at right.



a) Find the lengths of the sides of the four squares shown.

b) Find an explicit formula for the length of a side of the n^{th} square.

c) Find an explicit formula for the sum of the areas of the first n squares.

In problems 46 to 52, determine whether the series is arithmetic or geometric. Then find its sum.

46. $\displaystyle\sum_{i=1}^{20}(3i-8)$

47. $\displaystyle\sum_{j=6}^{50}\frac{1}{2}j$

48. $\displaystyle\sum_{k=1}^{6}10\left(\frac{1}{2}\right)^k$

49. $\displaystyle\sum_{n=1}^{\infty}0.2\left(\frac{1}{10}\right)^{n-1}$

50. $\displaystyle\sum_{n=1}^{\infty}20\left(\frac{1}{10}\right)^n$

51. $\displaystyle\sum_{k=1}^{200}(2k+3)$

52. $\displaystyle\sum_{i=-1}^{\infty}\frac{1}{2}\left(\frac{3}{10}\right)^i$

In problems 53 to 58, find the first term in the series expansion and determine whether the series has a finite sum or not. If it does, set up, but do not evaluate an expression which will give the sum. If the sum of the series is not finite, describe how you know this.

53. $\displaystyle\sum_{n=-2}^{\infty}20\left(\frac{2}{3}\right)^n$

54. $\displaystyle\sum_{m=1}^{\infty}\frac{1}{2}\left(-\frac{5}{4}\right)^{m-1}$

55. $\displaystyle\sum_{p=0}^{3}\frac{1}{9}\left(\frac{3}{2}\right)^{p+2}$

56. $\displaystyle\sum_{m=-1}^{1}-9\left(\frac{4}{3}\right)^{m+1}$

57. $\displaystyle\sum_{n=3}^{\infty}\frac{7}{6}\left(-\frac{1}{2}\right)^n$

58. $\displaystyle\sum_{p=0}^{\infty}\frac{1}{10}\cdot 2^{p-1}$

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59. Find the first three terms in the series and an expression for the n^{th} term in the series when the fraction $\frac{1}{3}$ is expressed as an infinite geometric series.

60. Find the first three terms in the series and an expression for the n^{th} term in the series when the fraction $\frac{2}{11}$ is expressed as an infinite geometric series.

61. If $p+7$, $3p$, and $2p+1$ are consecutive terms of an arithmetic sequence, find the next term.

62. If xy, x, 5, and y are the first four terms of a geometric sequence, find the values of x and y.

63. In an arithmetic sequence of complex numbers, $a_1 = 2-i$ and $a_2 = 5+i$. Find a_{25} and S_{25}.

64. In a geometric sequence of complex numbers, $a_1 = i$ and $r = 2i$. Find a_6 and S_6 using the formulas for geometric sequences and series.

65. State a formula for the sum of the first k positive integers.

66. Using the formula you wrote down in Problem 65, prove that if k is odd, the sum of *any* k consecutive integers (not just the first ones) will always be divisible by k. Try using induction.

67. Let $\overrightarrow{d} = (2,-1)$ be the common difference for an arithmetic vector sequence $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \ldots, \overrightarrow{a_n}$. If $\overrightarrow{a_1} = (3,0)$, find the equation for the line on which the terminal ends of $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \ldots, \overrightarrow{a_n}$ lie.

68. Evaluate $\displaystyle\sum_{n=1}^{\infty}1.89(0.97)^{n-1}$.

69. Consider an infinite geometric series of positive terms that sums to some positive number S. Determine the sum of the odd-numbered terms of the series in terms of r, the common ratio of the series.

70. Let $a_1, a_2, a_3, \ldots, a_n$ be an arithmetic sequence. Prove that $a_2, a_4, a_6, \ldots, a_{2n}$ is also an arithmetic sequence. Hint: show the new sequence has the characteristics of an arithmetic sequence.

71. Let a_n be the arithmetic sequence 4, 11, 18, 25, 32, ..., and let b_n be the arithmetic sequence 2, 8, 14, 20, ... Let c_n represent the terms that are in both a_n and b_n. Show that c_n is an arithmetic sequence.

72. Let a_n be an arithmetic sequence of positive integers with a positive common difference. Prove that if b_n is an arithmetic sequence, then b_{a_n} is an arithmetic sequence.

73. Find a *non-recursive* formula for the apparent nth term of the sequence defined by $a_1 = 2$ and $a_{n+1} = 2(a_n) - 1$.

74. Find a *non-recursive* formula for the sequence defined as $a_1 = 2$ and $a_n = a_{n-1} + 2(n-1)$.

75. Find a *recursive* formula for the sequence a_n: 1, 3, 6, 10, 15, ...

76. Using a_n from Problem 75, find a *recursive* formula for the sequence b_n: a_1^2, a_2^2, a_3^2, ...

77. Consider the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots$. Write the appropriate sigma notation for this series.

78. Evaluate the sum of the series in Problem 77. Hint: break each fraction into two fractions using the given factoring of the denominator.

79. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cdot(n+2)}$.

80. Let a_n be an arithmetic series of positive numbers with a positive common difference d. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n \cdot a_{n+1}}$.

81. Write an expression for the apparent nth term of the following sequences.

a) $1, 3, 1, 3, 1, \ldots$

b) $2, 1, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \frac{7}{11}, \ldots$

82. Write the first 7 terms of the sequence defined recursively as follows: $a_1 = 1, a_{k+1} = \dfrac{2}{a_k} + 1$. Then, come up with a non-recursive definition of this sequence.

83. Suppose there are k terms in an arithmetic sequence such that $a_1 = 1$ and $a_k = 61$. Give a formula for the sequence d_n of common differences that result as k increases from 2 to n. For example, if $k = 2$, then $d_2 = 60$, since $a_k = a_2$ and $a_2 - a_1 = 60$. If $k = 3$, then $a_k = a_3 = a_1 + (3-1)d_3$, by which you can solve for d_3. Find the pattern, then state the formula for the apparent nth term of the sequence d_n.

84. Repeat Problem 83, except consider a geometric sequence where $a_k = 2^{12} = 4096$, and you want to find a formula for the apparent nth term of the sequence of common ratios r_n.

85. If $A_3 = \begin{bmatrix} 7 & 1 \\ 2 & 8 \end{bmatrix}$ and $A_5 = \begin{bmatrix} 3 & 13 \\ 12 & -10 \end{bmatrix}$ in an arithmetic sequence, determine A_{10}.

86. Prove or disprove: If a_n is an arithmetic sequence, then 2^{a_n} is a geometric sequence (a_n is the exponent)

Section 1.3 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

Problem Set 1.3

1. Write two different equations in *point-slope* form for the line of slope -2 which passes through the point $(-5, 9)$.

In problems 2 to 8, evaluate the expression without a calculator.

2. $\left(\dfrac{8}{27}\right)^{-1/3}$
3. $\dfrac{2^{-1}}{32^{2/5}}$
4. $\log_{16} 8$
5. $\log_{0.2} \sqrt[3]{25}$
6. $\log(-100)$
7. $\log_{\sqrt{3}}\left(\dfrac{1}{9}\right)$
8. $\displaystyle\sum_{n=1}^{\infty} 0.6\left(\frac{1}{10}\right)^{n-1}$

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9. If $\log P = \frac{1}{2}\log 36 - 3\log 2x + \log\left(\frac{1}{3}\right)$, find P in simplest form.
10. Find the exact values of the x- and y-intercepts of the graph of $y = 3^x - 160$, or state that they do not exist. If any intercept is irrational, in addition to giving its exact value state the two consecutive integers it falls between.
11. Simplify completely: $\dfrac{\dfrac{1}{x-3} - \dfrac{2}{3x}}{\dfrac{2}{x} - \dfrac{2}{x+4}}$
12. Solve (without a calculator): $2\log_2(x+3) - \log_2(x+2) = 2$
13. Find the value of $\displaystyle\sum_{n=3}^{84} i^n$.
14. Find S_4 for $\displaystyle\sum_{i=2}^{\infty} \frac{12}{i}$.
15. State the formula for finding the sum of the first n terms of a geometric sequence and show how to use the formula to find the sum of the first 7 terms of the geometric sequence $128i$, -64 ,$-32i$, . . .
16. Is $\displaystyle\sum_{i=2}^{20} 7$ an arithmetic series, a geometric series, or neither? Explain.

2 Function Fundamentals

Section 2.1 Key vocabulary, notation, and operations

Definition 2.1.1 A **function** is a rule that produces exactly one for each allowable input.

Any way you can think of for describing a rule that produces exactly one output for each allowable input is a way of specifying a function. Some possibilities are a sentence, an equation, a graph, and a table of values.

Think About It 2.1.1 Tell whether the rule describes a function. If it does not, use the definition of a function along with some key inputs and outputs to explain how you know. If it does, write down what you'd say to convince someone who thinks it doesn't.

Rule A:

input	4	0	2	-1	7
output	2	-1	5	-4	-4

Rule B:

input	2	0	5	-4	-4
output	4	0	2	-1	7

Rule C: The input is a number and the output is what you get when you divide 24 by the input

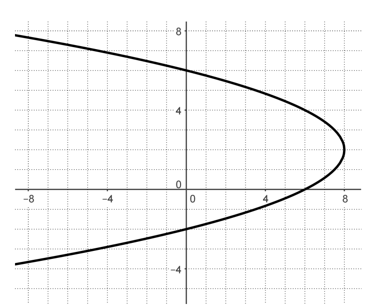
Rule D: The input is the full name of one of the first 45 presidents of the U.S. and the output is the first name of a child of the input.

Rule E: The input is the full name of one of the first 45 presidents of the U.S. and the output is the gender of the input.

Rule F: the input is a value that could be substituted for a in the equation $a^2 + b^2 = 25$ and the output is a value of b that would make the equation true.

Rule G: The input is the x-coordinate of any point on the graph at right and the output is the y-coordinate of that point

Rule H: The input is the y-coordinate of any point on the graph at right and the output is the x-coordinate of that point



To avoid having to say things like "the input is" and "the output is" or "find y when $x =$" , we use the $f(x)$ notation developed by Leonhard Euler (of Section 1.2 fame). *This notation makes it easy to show both the input and output compactly.* For example, the statement $f(3) = 7$ tells us that the output of the function f is 7 when the input is 3, as would be the case, for example, if the function f were defined by the equation $f(x) = x^2 - 2$. We would say in this example that f **is a function of** x, which means that the value of f depends on the value of x.

In representing functions graphically, it is traditional (though not absolutely necessary) to plot the inputs on the horizontal axis and the outputs on the vertical axis. While we often use the letter x to represent the input of a function, it is certainly not necessary to do so. The equations $Q(t) = t^2 - 2$ and $\lambda(\clubsuit) = \clubsuit^2 - 2$ represent exactly the same function as $f(x) = x^2 - 2$, and the graphs of all three equations are identical.

Think About It 2.1.2 You have likely heard of the **vertical line test** in your previous study of functions, and you may recall having applied this test by considering whether it is possible to draw a vertical line somewhere that will cross a given graph more than once. If it is possible, then you say that the graph "fails the vertical line test" and conclude, that the graph is *not* the graph of a function. Explain how this test comes from Definition 2.1.1 and identify the important assumption are you making about inputs and outputs in coming to the conclusion that a graph which fails the vertical line test is not the graph of a function. What would be a more precise conclusion?

One of the things that helps you to get to know a particular function is figuring out what values can be put into the function and what values can come out:

Definition 2.1.2 The **domain** of a function is the set of all allowable inputs to the function.

Definition 2.1.3 The **range** of a function is the set of all possible outputs of a function.

Of course, this begs the question, what makes an input "allowable"?

The answer is, well, it depends on the context. Think again about Rule E in TAI 2.1.1. For this function, *John Quincy Adams* is an allowable input, but *Barack*, *James Knox Polk*, and *Alexander Hamilton* are not, each for a different reason. (What are the reasons?). **In this course, when working with functions whose inputs and outputs are numerical, we'll typically allow all inputs which produce real numbers as outputs.** Sometimes, however, this won't make sense. For example, what is the domain of $A(x) = x^2$? What if we're using $A(x)$ to represent the area of square which has a side of length x? (And, sometimes, if you're feeling adventurous, you might decide you'd like to allow imaginary numbers as outputs. Or as inputs.[1])

[1]Feeling adventurous, huh? Consider the function $f(x) = \sqrt{x}$. How is the domain different if you do or don't allow complex numbers as outputs? And what happens if you allow complex numbers as inputs? Ask yourself, "Can I find a value for $f(i)$?" If

It's worth being aware that it is possible to include a restriction on the domain as part of the definition of a particular function. Here's an example: $f(x) = (x-3)^2, x < 3$. The graph of this function is as close as you can get to half of a parabola.

When trying to determine the domain of a function, it's often useful to ask yourself what inputs will lead to trouble. For example, any number that, if used as an input would make a denominator zero is excluded from the function's domain. Similarly, numbers that, if used as inputs, would result in having to take the square root of a negative number or having to take a log of zero or a negative number will not be part of a function's domain.

Think About It 2.1.3 What is the domain of each function?

$$f(x) = \frac{x+1}{x-4}$$

$$g(x) = \sqrt{x-4}$$

$$h(x) = \log(x-4)$$

$$k(x) = (x-4)^2$$

Think About It 2.1.4 Find a function whose domain is $\{x|x < 10\}$.

Think About It 2.1.5 Since they are sets, domains are often expressed using set-builder notation as was done in TAI 2.1.4. Interval notation, discussed below, is even more common. Another alternative is to write a very precise phrase, such as "all real numbers less than 10." It is also acceptable to write $x < 10$. It is NOT, however, acceptable to state the domain by writing $\mathbb{R} < 10$. What makes this last form problematic?

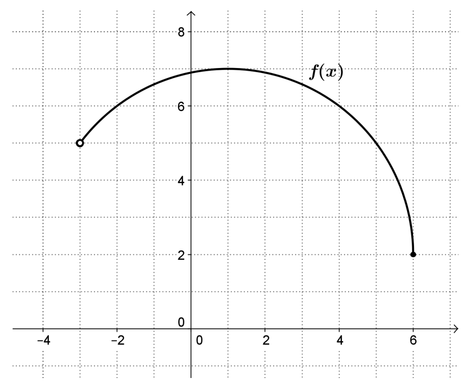
Determining the range of a function typically[2] requires quite a bit more effort than finding the domain. Knowing what a few very important functions look like along with knowing how tweaks to the equations of these important functions affect their graphs vastly reduces the effort in many cases. Further developing such knowledge is one focus of this course. Calculus is a huge help, too. That's for the next course. Using technology to graph the function is certainly helpful as well, as long as you have a good sense of what you should be paying particular attention to.

you've thought about this a bit and aren't sure where to begin, try assuming that the output will be complex, *i.e.*, that it will have the form $a + bi$. Think about what you might do next. If you get stuck, perhaps this will help: $(a+bi)^2 = a^2 + 2abi + b^2i^2 = (a^2 - b^2) + (2ab)i$. Or perhaps not. Keep thinking. Remember the Grinch, who "puzzled and puzzled 'till his puzzler was sore. Then the Grinch thought of something he hadn't before."

[2]but not always; see, for example, Rule E in TAI 2.1.1.

Think About It 2.1.6 Figure out what you'll need to pay particular attention to and then use Desmos or some other graphing technology to help you find the domain and range of $f(x) = \dfrac{(x+2)(x-5)(x+1)}{x^2-8x-9}$. Sketch a neatly labeled graph to support your answer.

•Interval Notation• A notation that comes in handy when specifying domain and range is **interval notation**. In this notation, we indicate a span of values by writing the lowest value followed by the highest value with a comma in between. If the lowest and highest values are to be *excluded* we surround the values with parentheses. If they are to be *included* we use brackets. We can use a parenthesis on one end and a bracket on the other if one end value is excluded and the other is included. Thus, for the function $f(x)$ graphed at right, the domain is $(-3, 6\,]$ and the range is $[\,2, 7\,]$.



To indicate that a domain or range is infinite in either or both directions, we include $-\infty$ and/or ∞ in the interval expression. Be aware that since ∞ is not a number it is never "included", and, thus, never written with a bracket. As an example, consider the domain of $g(x) = \sqrt{x}$, which is $[\,0, \infty)$

An interval in which the endpoints are included, such as $[0, 10]$, is called a **closed interval**, while an interval which does not include the endpoints, such as $(0, 10)$ is called an **open interval**. Intervals which include one endpoint, but not the other, like the domain of $g(x) = \sqrt{x}$, are referred to interchangeably as either **half-open** or **half-closed**.

Think About It 2.1.7 Write the domain of each function in interval notation and then classify the domain as an open interval, a closed interval, a half-open interval, or some combination of these.

$A(x) = x^2 - 3$

$E(p) = \sqrt[3]{9 - p^2}$

$B(x) = 3$

$F(\alpha) = \sqrt{\alpha^2 - 9}$

$C(t) = \sqrt{t + 3}$

$G(w) = \log_3 w$

$D(t) = \sqrt[3]{t + 3}$

$H(w) = \dfrac{1}{\sqrt{3 - w}}$

•Reflecting on what it means to "find the solution"• Since you began studying algebra, you've frequently been told to "solve" or "find the solution," and, over time, you likely have developed the often-useful habit of proceeding to do so by following a sequence of algebraic steps. Given this history, you shouldn't be to hard on yourself if you've come to believe–incorrectly–that "find the solution" and "solve" actually mean "do some algebra." Take a minute to think back to the days of "guess and check." You knew then that what "solve" actually meant was "find the value(s) of the variable(s) that make the equation or inequality true." The technique you use does not necessarily have to involve algebraic manipulation. Remembering this is helpful when you're confronted with a problem like the one in Example 2.1.1.

Example 2.1.1 The graph of a function $f(x)$ is shown at right. Use the graph to find the solutions to the given equation or inequality.

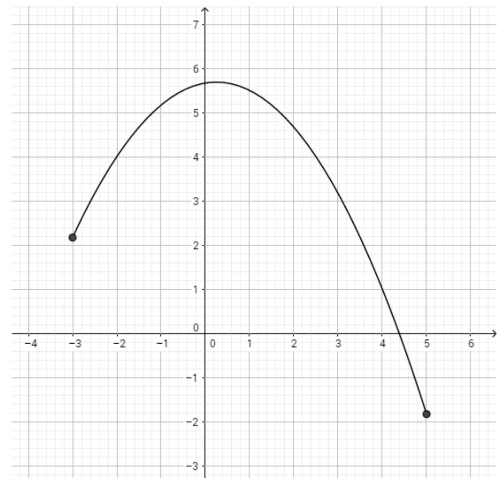
a) $f(x) = 1$

b) $f(x) = -\frac{1}{2}(x+4)+5$

c) $f(x) = x^2$

d) $f(2x) = 4$

e) $f(x) < 4$



Solution First, it's crucial to note that the equations to be solved are NOT new definitions of f. As you were told at the outset, f is defined by the given graph.

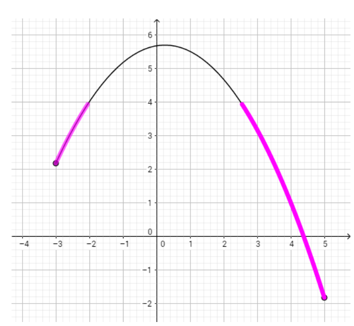
a) The values of x that will make this equation true are those inputs to the function f which result in an output of 1. The outputs of f are its y-coordinates, and the only point on the graph of f whose y-coordinate is 1 is the point $(4, 1)$. **This tells us that $x = 4$ is the solution to the given equation.**

b) Before tackling this one, it's useful to note than an alternative way of solving part (a) would have been to graph the horizontal line $y = 1$ and look for where it crossed the graph of f. With that in mind, we think of the right-hand side of the equation we're given this time as a function, $g(x)$, and look for where the graphs of f and g intersect. This tells us which inputs to f and g which produce identical outputs and, therefore, are the solutions to the equation $f(x) = g(x)$. The graph of our newly-minted g is a line with a slope of $-\frac{1}{2}$ passing through the point $(-4, 5)$. This line intersects f at the points $(-2, 4)$ and $(4, 1)$, **so the solutions are $x = -2$ and $x = 4$.**

c) By carefully graphing $g(x) = x^2$, we see that the graphs of f and g intersect at $(-2, 4)$ and at the point whose coordinates are approximately $(2.1, 4.5)$, **so the solutions are $x = -2$ and $x \approx 2.1$.** (To find the exact value of the second solution in this case, we would need to determine the equation that produces the graph of f and solve algebraically.)

d) Since we are interested here in inputs that produce an output of 4, we can draw the horizontal line $y = 4$ to determine that inputs to f of -2 and 2.5 produce the desired output. In this case, however, we must further note that the inputs to f are not x, but are $2x$. This means that we need x values equal to half of -2 and 2.5 to produce the desired inputs (and, consequently, the desired output), **so the solutions are $x = -1$ and $x = 1.25$.**

e) Here we want to find the x-coordinates of the points on the graph of f where the y-coordinate is less than 5. These portions of the graph have been shaded in the image at right. From this picture we can see that **the solution to this inequality can be expressed in interval notation as $[-3, -2) \cup (2.5, 5]$.**



•Operations on Functions• Functions can be combined in a variety of ways to make new functions. You can, for example, combine them using the four operations of arithmetic, just as you can combine numbers. You'll want to be aware that the notation $f(x)+g(x)$ can also be written as $(f+g)(x)$ or even just $f+g$. The same holds true for the other arithmetic operations which means that $f(x)\cdot g(x)$ can be written as $(fg)(x)$ or as fg.

Example 2.1.2 If $f(x)=x^2+5x$ and $g(x)=x-3$, find

a) $f(x)+g(x)$ b) $(f/g)(x)$ c) fg

Solution

a) $$f(x)+g(x)=(x^2+5x)+(x-3) \\ =x^2+6x-3$$

b) $$(f/g)(x)=\frac{f(x)}{g(x)}=\frac{x^2+5x}{x-3}$$

c) $$\begin{aligned} fg &= f(x)\cdot g(x) \\ &= (x^2+5x)(x-3) \\ &= x^3-3x^2+5x^2-15x \\ &= x^3+2x^2-15x \end{aligned}$$

Another way of combining functions is using an operation known as **composition**. Composition is what you're doing if, instead of using a number or a variable as an input to a function, you use another function as the input. The result is a whole new function. For a concrete example of composition, you might think of a waiter's daily income as function, f, of the number of customers who come into the restaurant that day, and then think about how the number of customers who come into a restaurant could be a function, g, of what day of the week it is. A composition of these functions would be the waiter's daily income as a function, h, of what day of the week it is.

Think About It 2.1.8 Give sample inputs and outputs for f, g, and h over the course of one week for the waiter-in-a-restaurant example in the preceding paragraph.

Example 2.1.3 If $f(x)=x^2+5x+1$, find a) $f(a+3)$ b) $f(4x)$

Solution

a) The expression $f(a+3)$ tells us that the input to our rule should be $a+3$ rather than x, so we replace each x in the statement of the rule by $a+3$:
$$\begin{aligned} f(a+3) &= (a+3)^2+5(a+3)+1 \\ &= (a^2+6a+9)+(5a+15)+1 \\ &= a^2+11a+25 \end{aligned}$$

b) The expression $f(4x)$ tells us that the input to our rule should be $4x$ rather than x, so we replace each x in the statement of the rule by $4x$:
$$\begin{aligned} f(4x) &= (4x)^2+5(4x)+1 \\ &= 4x^2+20x+1 \end{aligned}$$

To indicate that we want to use the function g as the input to the function f we can write $f(g(x))$ or $f\circ g$

or $(f \circ g)(x)$. The first version makes it really clear that we're supposed to replace x with function $g(x)$ wherever it occurs in the definition of $f(x)$, but all of the notations mean the same thing. Note that $\circ$ does **not** signify multiplication.

Example 2.1.4 If $f(x) = x^2 + 5x$ and $g(x) = x - 3$, find a) $f \circ g$ b) $(g \circ f)(3x + 2)$

Solution

a)
$$\begin{aligned} f \circ g &= f(g(x)) \\ &= f(x-3) \\ &= (x-3)^2 + 5(x-3) \\ &= (x^2 - 6x + 9) + (5x - 15) \\ &= x^2 - x - 6 \end{aligned}$$

b) $(g \circ f)(3x + 2) = g(f(3x + 2))$

We can find $f(3x + 2)$ first and then use our result as the input to g.

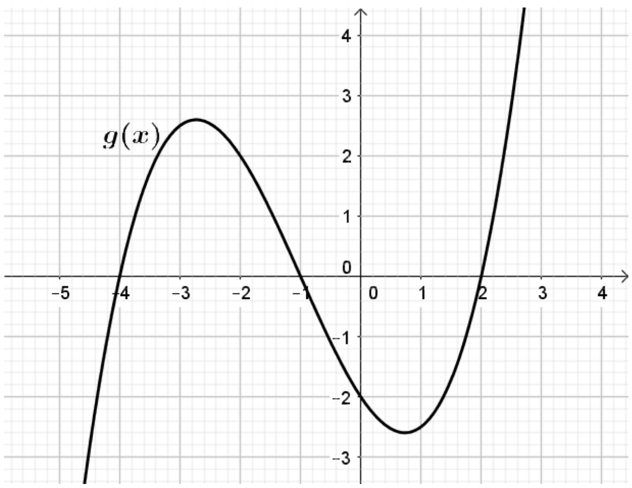
$$\begin{aligned} f(3x+2) &= (3x+2)^2 + 5(3x+2) \\ &= (9x^2 + 12x + 4) + (15x + 10) \\ &= 9x^2 + 27x + 14 \end{aligned}$$

So,
$$\begin{aligned} g(f(3x+2)) &= g(9x^2 + 27x + 14) \\ &= (9x^2 + 27x + 14) - 3 \\ &= 9x^2 + 27x + 11 \end{aligned}$$

Think About It 2.1.9 Consider the functions $f(x) = \sqrt{x}$, $g(x) = x^2 - 9$, and $h(x) = f \circ g$. Find $h(-5)$, $h(1)$, $h(7)$, and state the domain of h. Sketch the graph of g and explain the connection between the graph of g and the domain of h.

Think About It 2.1.10 Consider the function $g(x)$ whose graph is shown below. State the solutions to

a) $g(x) = 0$ b) $g(x) < 0$ c) $g(3x) = 0$



d) Use the thinking you've done about these solutions to help you determine the domains of the following functions.

$$h(x) = \sqrt{g(x)}$$

$$j(x) = \frac{1}{g(x)}$$

$$k(x) = \frac{1}{g(3x)}$$

Hint: In thinking about the domain of each of these, keep in mind that your ultimate goal is to find the allowable values of x. To determine these for $h(x)$, you first need to realize that when $g(x)$ is negative, $h(x)$ will be undefined. In part (b) of this TAI you thought about which x values resulted in negative $g(x)$ values. These are the x values that are excluded from the domain of h. You can see the solution to a similar problem here: www.geogebra.org/m/ackHpxG4

Think About It 2.1.11 Consider the situation where f and g are both linear functions and you combine them either by composition or by using an arithmetical operation. Which ways of combining them will always produce a linear function? Which will sometimes produce a linear function? Which will never produce a linear function?

Problem Set 2.1

1. The domain of a particular function $f(x)$ is $\{x \mid x \leq 5\}$ and the range of the function is $\{x \mid x > 0\}$ Write the domain and range in interval notation.

2. The domain of a particular function $g(x)$ is $\{x \mid x \geq -3\}$ and the range of the function is $\{x \mid x \leq 6\}$ Write the domain and range in interval notation.

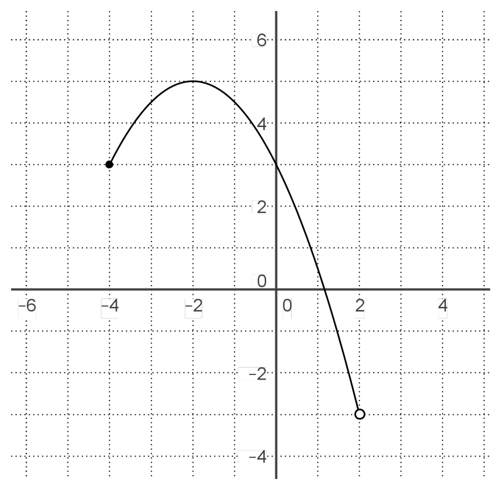
3. The domain of a particular function $h(x)$ is $\{x \mid \ 7 \geq x > -3\}$ and the range of the function is $\{x \mid x \leq 2\}$ Write the domain and the range in interval notation.

4. Give an example (in the form of an equation) of a constant function and state its domain and range.

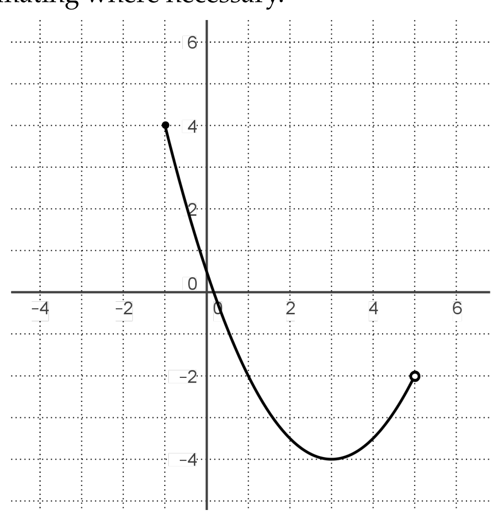
5. Write an equation for the constant function $f(x)$ which passes through the point $(3, -1)$ and state its domain and range using interval notation.

6. If $f(x) = x^2 - x + 1$ and $g(x) = 2x - 3$, find
   a) $(f \circ g)(-1)$ b) $(f \circ g)(x)$ c) $(g \circ f)(x)$ d) $(g \circ g)(x)$

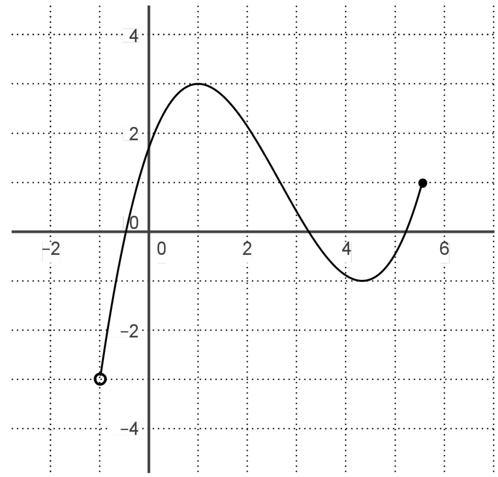
7. The function $f(x)$ is graphed at right. Find the following, estimating where necessary.
   a) the domain and range of f
   b) the solution(s) to the equation $f(x) = 0$
   c) the solution(s) to the equation $f(x) = -\dfrac{5}{2}x$
   d) the solution to the inequality $f(x) \leq 3$
   e) $(f \circ f)(1.5)$
   f) the domain of $h(x) = \sqrt{f(x)}$
   g) the domain of $h(x) = \sqrt{-f(x)}$
   h) the domain of $h(x) = \dfrac{1}{f(x)}$
   i) the domain of $h(x) = -f(x)$
   j) the domain of $h(x) = \log f(x)$



8. The function $f(x)$ is graphed at right. Find the following, estimating where necessary.
   a) the domain and range of f
   b) the solution(s) to the equation $f(x) = -2$
   c) the solution(s) to the equation $f(x) = -x - 1$
   d) the solution to the inequality $f(x) > -3.5$
   e) $(f \circ f)(-1)$
   f) the domain of $h(x) = \log(f(x))$
   g) the domain of $h(x) = \sqrt{-f(x)}$
   h) the domain of $h(x) = \sqrt[3]{f(x)}$
   i) the domain of $h(x) = \sqrt{f(x) - 4}$
   j) the domain of $h(x) = \dfrac{-1}{f(x)}$



9. The function $f(x)$ is graphed at right. Find the following, estimating where necessary.



a) the domain and range of f

b) the solution(s) to the equation $f(x) = 0$

c) the solution(s) to the equation $f(x) = x$

d) the solution(s) to the equation $f(x) = x^2$

e) the solution(s) to the equation $f(x) = -\frac{2}{5}(x-6) - 1$

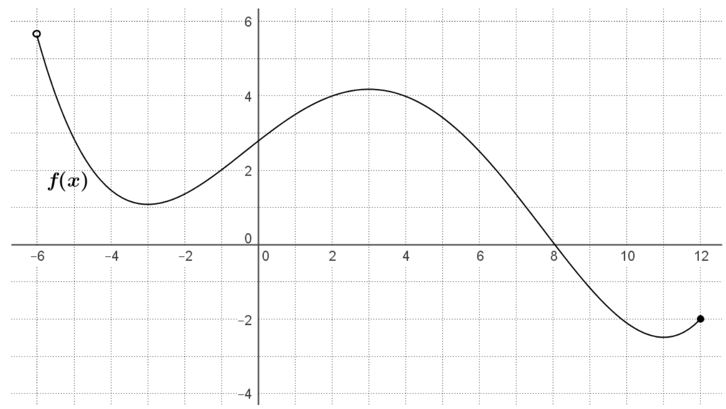
f) the solution to the inequality $f(x) \leq -1$

g) $(f \circ f)(1)$

h) the domain of $h(x) = \dfrac{1}{f(x)}$

i) the domain of $h(x) = 2\sqrt{-x}$

10. The function $f(x)$ is graphed at right. Find the following, estimating where necessary.



a) the domain and range of f

b) the solution(s) to $f(x) = 0$

c) the solution(s) to the equation $f(x) = -\frac{3}{4}(x-8) + 1$

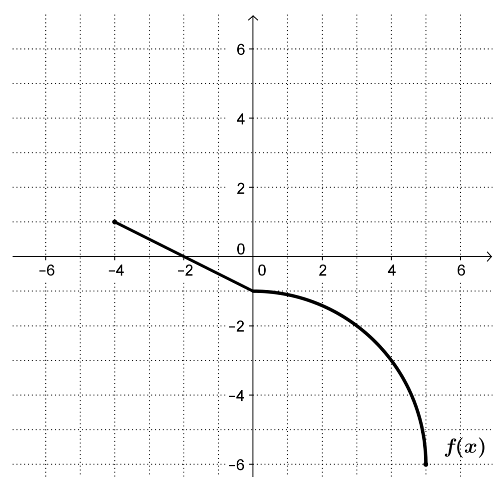
d) the solution to the inequality $f(x) < 1$

e) $(f \circ f)(-1)$

f) the domain of $g(x) = \sqrt{f(x)}$

g) the domain of $g(x) = \dfrac{1}{2x}$

11. If f is the function graphed at right and $g(x) = x^2 - 5$, find the following, estimating where necessary.



a) $f(g(-2))$

b) $(g \circ f)(-2)$

c) $f(f(4))$

d) $(g \circ g)(3)$

12. A pot containing 800 ml of boiling water is removed from the stove and left to sit on the counter. After 2 hours someone comes along, wonders why the pot was left on the counter, and pours the water out. The graph of $f(t)$ at right shows the temperature of the water as a function of the length of time it has been sitting on the counter. Use this information to answer the questions that follow, estimating where necessary.

a) What are the units of t?

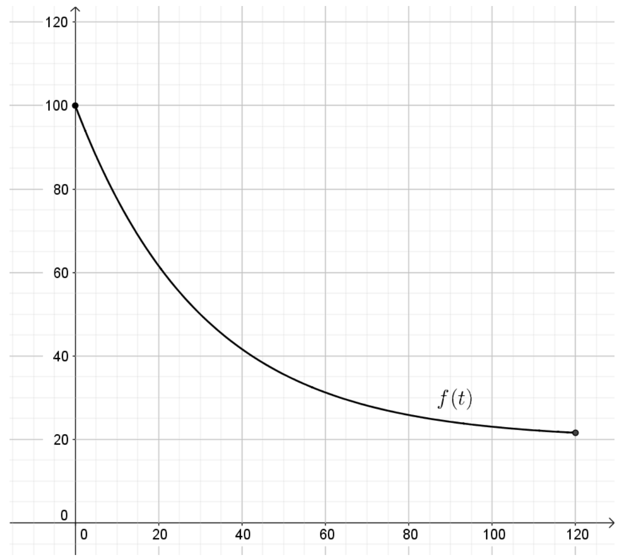
b) What are the units of f?

c) State the domain and range of f using interval notation.

d) State the solution to the equation $f(t) = 50$. In this context, what question is this solution the answer to?

e) State the solution to the inequality $f(t) < 25$. In this context, what question is this solution the answer to?

f) What is the value (including units) of the expression $\dfrac{f(85) - f(25)}{85 - 25}$. In this context, what question is this value the answer to?



In problems 13 to 28,

a) State the domain of h.

b) Identify two functions f and g, such that $f \circ g = h$. Then find $g \circ f$. (Do not use $f(x) = x$ or $g(x) = x$. Why?)

13. $h(x) = x^2 + 4$

14. $h(x) = (x+1)^3$

15. $h(x) = \dfrac{4}{\sqrt{x} - 5}$

16. $h(x) = \dfrac{1}{\sqrt{x - 5}}$

17. $h(x) = \dfrac{1}{x} - 8$

18. $h(x) = \sqrt{3 - x}$

19. $h(x) = 5 - \frac{1}{2}x^3$

20. $h(x) = 3^{x+5}$

21. $h(x) = 6 + 2^x$

22. $h(x) = x^2 + 6x + 9$

23. $h(x) = \dfrac{3}{5x^2}$

24. $h(x) = \dfrac{-1}{x^2 - 5}$

25. $h(x) = \dfrac{3}{2x + 9}$

26. $h(x) = \dfrac{2x}{6x - 1}$

27. $h(x) = \sqrt{x^2 - 9}$

28. $h(x) = \sqrt{16 - x^2}$

---

29. Find $f \circ g \circ h$ if $f(x) = \sqrt{x}$, $g(x) = \dfrac{x-1}{2x-3}$, and $h(x) = 3x + 1$.

30. Find $h \circ g \circ f$ if $f(x) = \sqrt{x}$, $g(x) = \dfrac{x-1}{2x-3}$, and $h(x) = 3x + 1$.

31. Find $f \circ g \circ h$ if $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{x}{3x-4}$, and $h(x) = x^2 - 1$.

32. Find $g \circ h \circ f$ if $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{x}{3x-4}$, and $h(x) = x^2 - 1$.

33. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f(g(x)) = h(x)$.

34. A square which initially has an area of 100 cm^2 shrinks in such a way that its area decreases by 5 cm^2 per minute.

a) Write an equation for a function f which gives the area of the square as function of the time (in minutes) since it started shrinking.
b) Write an equation for a function g which gives the length of the side of the square as function of the square's area.
c) If $h = g \circ f$, find h and indicate precisely what information about the situation this provides.

35. A circle which initially has an area of 36π cm^2 expands in such a way that its area increases by 2π cm^2 per minute.

a) Write an equation for a function f which gives the area of the circle as function of the time (in minutes) since it started expanding.
b) Write an equation for a function g which gives the radius of the circle as function of the circle's area.
c) If $h = g \circ f$, find h and indicate precisely what information about the situation this provides.

Section 2.2 End Behavior and Limits

As we seek to fully understand a particular function, one fundamental question, which we explored in Section 2.1, is what values can go in and what values can come out. Before we can say we know a function well, however, there are many other features we need to think about. A few such features are *intercepts*, which we considered in Section 1.1, *extreme values*, which we will consider in Section 2.3, and *end behavior*, a focus of this section.

End behavior, informally, is what happens to the values of the function, *i.e.*, the output values, as the input values head toward the ends of the domain. Do the values of the function head toward ∞ or $-\infty$? Or do they eventually get closer and closer to some particular number? Or do they oscillate between two values? Or . . . ?

One form of mathematical notation that we can use to describe end behavior is **limit notation**, which is illustrated in the following example.

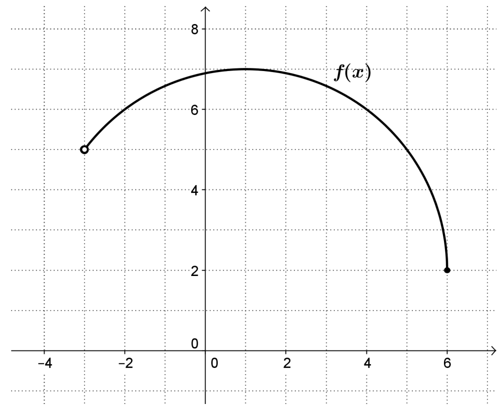
Example 2.2.1 State the domain of $f(x) = x^2$ and use limit notation to describe the end behavior of f.

Solution Since we can square any number and get a real number as a result, the domain is $(-\infty, \infty)$.

To describe the behavior towards the left "end" of the function, we observe that as x takes on large negative values, the function takes on large positive values and we can make the output as large as we like by making the input sufficiently large, so we write $\lim\limits_{x \to -\infty} f(x) = \infty$. (When reading this aloud we say, "The limit of f as x approaches negative infinity is infinity.")

To describe the behavior towards the right "end" of the function, we observe that as x takes on large positive values, we can again make the output as large as we like by making the input sufficiently large, so we write $\lim\limits_{x \to \infty} f(x) = \infty$.

Example 2.2.2 In Section 2.1 we encountered the function f shown below and determined that its domain was $(-3, 6]$. Use limit notation to describe the end behavior of f.



Solution In this case, we want to write limit expressions that describe what happens to the function value as x approaches -3 and as x approaches 6. We see that when we're on the function in the vicinity of $x = -3$ and we consider x-values closer and closer to -3, the value of the function gets closer and closer to 5, so we write $\lim\limits_{x\to -3^+} f(x) = 5$. The ″+″ superscript following the -3 is what we use in limit notation to indicate that we're considering the case where x is approaching -3 from values larger than -3. To describe the end behavior at the right end of the domain, we write $\lim\limits_{x\to 6^-} f(x) = 2$.

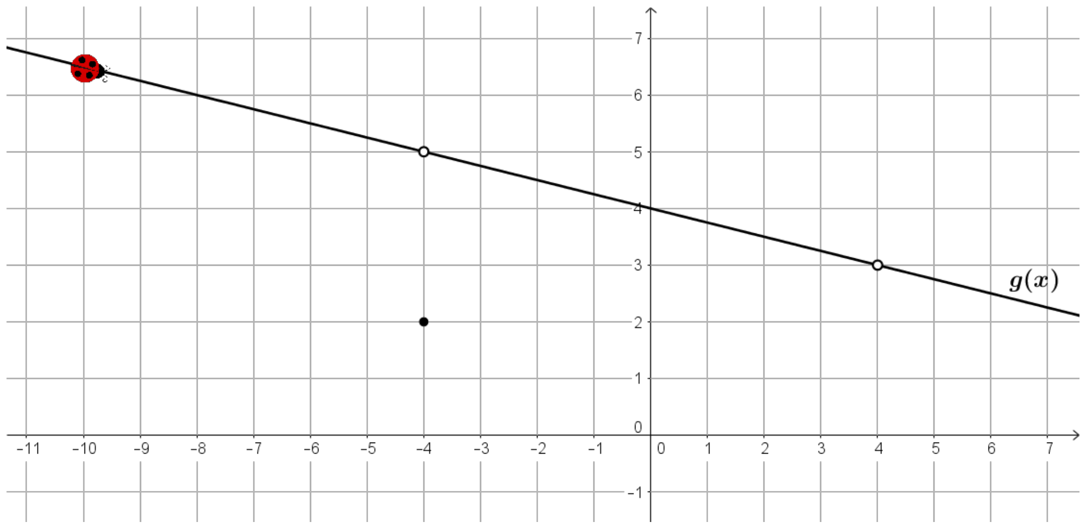
It's worth noting that it makes no difference in specifying the limit whether the function is defined at the value which x approaches or not.

(Take note of the spacing in a limit expression for when you write one yourself. Everything is on the same line except the "$x \to a$" piece which fits neatly under the "lim" piece, extending to a bit to each side if necessary.)

Think About It 2.2.1 Describe the end behavior of the functions $f(x) = -2x^3$, $g(x) = \left(\dfrac{1}{2}\right)^x$, and $h(x) = 2+\sqrt{x-5}$ using limit notation. Write equations or draw graphs of some other functions and use limit notation to describe the end behavior of your functions.

Limit notation is not limited (☺) to describing end behavior. Speaking very informally, limit notation gives us a way to to describe what we would expect the output of a function to be for a particular input if, while traveling along the function, we could get as close to that input value as we wanted, but couldn't actually see the function at the input value (or beyond). Most of the time, of course, the output is just what we'd expect. Occasionally, however, a function is not so well behaved.

Imagine you are the ladybug crawling along the function g shown below.



For most of your journey, the outputs are just what you expect them to be. As you approach the point where the input is -8 (remember to imagine that you can't actually see the function at $x = -8$ or for any larger x-values), you're expecting that the output will be 6, so you can write

$$\lim_{x \to -8^-} g(x) = 6$$

The fact that the value of the function turns out to be 6 comes as no surprise. (Recall from Example 2.2.2 that the "$-$" superscript on the -8 means that you were describing what you were expecting as you approached $x = -8$ from the left.) And, continuing along your merry way, you begin to think about what you expect the function's output to be be when the input is -4. Since you're expecting it to be 5, you can write

$$\lim_{x \to -4^-} g(x) = 5$$

In this case, the function's actual value, 2, turns out to be a surprise. The value of the limit as x approaches -4 is *different* than the value of the function at $x = -4$, and that's just fine.
You continue along for another stretch where all the function's outputs are just what you're anticipating. The last surprise comes when the input is 4. After thinking about what you expect the output to be for an input of 4, you write

$$\lim_{x \to 4^-} g(x) = 3$$

You come to discover that $g(4)$ is actually undefined. Knowing that, however, doesn't change the value of the limit. The three limits above are called **one-sided limits** because when we wrote them we only considered our expectations as we approached various input values from one side. Another bug traveling in the opposite direction, could have written the corresponding one-sided limits with superscripts of "$+$":

$$\lim_{x \to 4^+} g(x) = 3, \quad \lim_{x \to -4^+} g(x) = 5, \quad \lim_{x \to -8^+} g(x) = 6$$

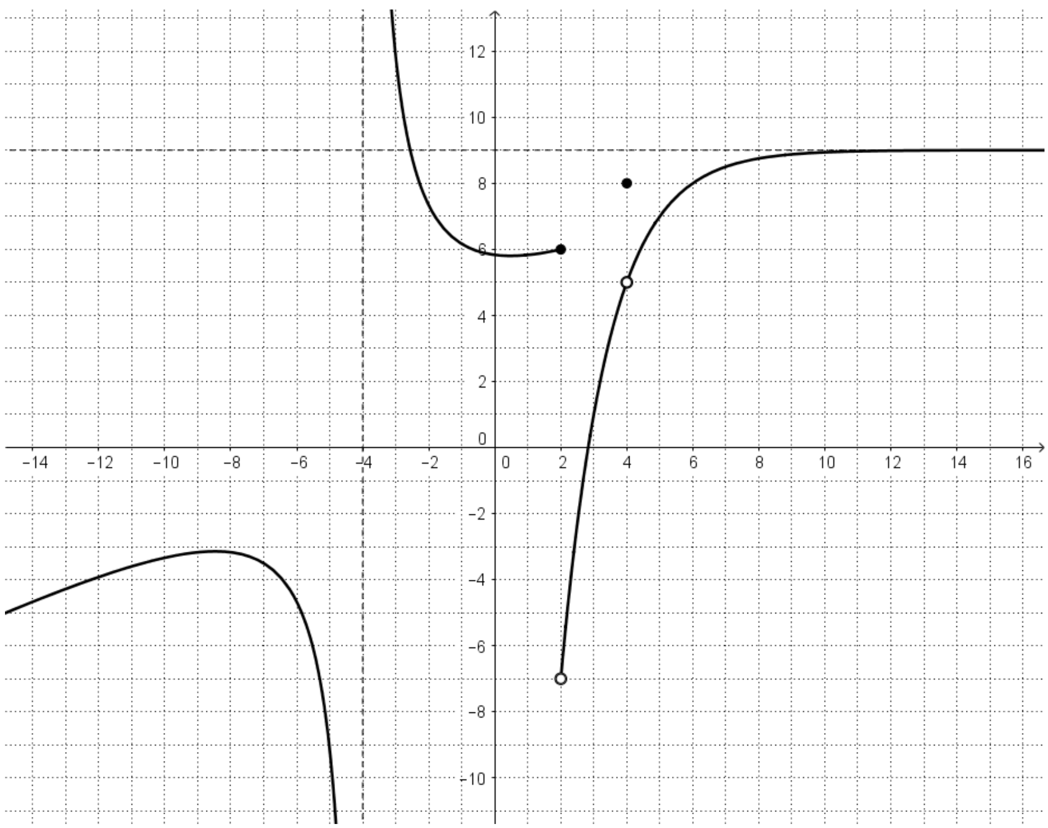
In each of these cases, the value of the limit is the same whether you approach the input value from the left or right. To indicate this, we can write what are known as **two-sided limits**:

$$\lim_{x \to 4} g(x) = 3, \quad \lim_{x \to -4} g(x) = 5, \quad \lim_{x \to -8} g(x) = 6$$

Think About It 2.2.2 Come up with an example of a function f where $\lim_{x\to 3^-} f(x)$ is different from $\lim_{x\to 3^+} f(x)$.

If we were asked to find the two-sided limit, $\lim_{x\to 3} f(x)$, in TAI 2.2.2 we would say that it doesn't exist.

Think About It 2.2.3 The crazy function, f, graphed below has a domain of all reals except -4. Look through the list of expressions and put a star next to each one that is a two-sided limit. Then use the graph to find the value of each expression, estimating as necessary.



a) $f(2)$

b) $\lim_{x\to 2^-} f(x)$

c) $\lim_{x\to 2^+} f(x)$

d) $\lim_{x\to 2} f(x)$

e) $f(3)$

f) $\lim_{x\to 3^-} f(x)$

g) $\lim_{x\to 3^+} f(x)$

h) $\lim_{x\to 3} f(x)$

i) $f(4)$

j) $\lim_{x\to 4^-} f(x)$

k) $\lim_{x\to 4^+} f(x)$

l) $\lim_{x\to 4} f(x)$

m) $f(-4)$

n) $\lim_{x\to -4^-} f(x)$

o) $\lim_{x\to -4^+} f(x)$

p) $\lim_{x\to -4} f(x)$

q) $\lim_{x\to \infty} f(x)$

r) $\lim_{x\to -\infty} f(x)$

Problem Set 2.2

1. Use limit notation to describe the end behavior of the function $f(x) = 5 - x^3$
2. Use limit notation to describe the end behavior of the function $f(x) = 2^x + 1$
3. Use limit notation to describe the end behavior of the function $f(x) = \left(\frac{1}{10}\right)^x - 2$

In problems 4 to 32

a) Determine the domain of the function without the help of technology.

b) After determining the domain, make a quick sketch of the function with the help of graphing technology and state the range. Label any values on the axes necessary to illustrate the domain and range.

c) Use your graph from part (b) to help you figure out how to describe the end behavior of the function and the behavior near any asymptote with limit notation.

4. $f(x) = x^2 - 5$
5. $g(x) = (x-5)^3$
6. $r(t) = (t-5)^{-1}$
7. $Q(x) = x^{1/2} - 5$
8. $j(x) = \dfrac{1}{x^{1/2} - 5}$
9. $k(x) = (x-5)^{1/2}$
10. $m(x) = \dfrac{1}{(x-5)^{1/2}}$
11. $R(t) = t^{1/3} - 5$
12. $f(x) = \dfrac{1}{x^{1/3} - 5}$
13. $g(t) = \sqrt{t+12}$
14. $h(z) = \dfrac{z+3}{z^2 - 7z + 10}$
15. $j(x) = \dfrac{2x}{\sqrt[5]{8-x}}$
16. $g(w) = \sqrt[4]{9-w}$
17. $p(r) = \dfrac{2r-4}{r^2 - 5r + 4}$
18. $C(w) = 5 - \frac{1}{2}w^3$
19. $h(z) = \sqrt[3]{(z+1)(z-4)}$
20. $k(z) = \sqrt{(z+1)(z-4)}$
21. $f(x) = \sqrt{(x+2)(x+11)}$
22. $f(t) = \sqrt{t^2 - 3t - 10}$
23. $f(t) = \sqrt{2t^2 - 3t + 1}$
24. $D(x) = \dfrac{x+2}{\sqrt{5-x}}$
25. $j(x) = \dfrac{\sqrt[3]{x-10}}{x+1}$
26. $P(x) = \dfrac{x+5}{\sqrt{(x+2)(x+11)}}$
27. $g(z) = \sqrt{z^2 - 5}$
28. $q(w) = \sqrt{2 - w^2}$
29. $f(x) = -\sqrt{9 - x^2}$
30. $h(t) = \sqrt{t^2 - 2t - 5}$
31. $h(t) = \sqrt{5 + 2t - t^2}$
32. $f(x) = -\dfrac{1}{x^2 - 4x - 21}$

The graph of a function $h(x)$ is shown below. It has a horizontal asymptote of $y = 2$ and a vertical asymptote of $x = 5$. Use this graph to find the limits in problems 33 to 43, estimating as necessary.

33. $\lim\limits_{x\to 2^-} h(x)$

34. $\lim\limits_{x\to 2^+} h(x)$

35. $\lim\limits_{x\to 2} h(x)$

36. $\lim\limits_{x\to 4^-} h(x)$

37. $\lim\limits_{x\to 4^+} h(x)$

38. $\lim\limits_{x\to 4} h(x)$

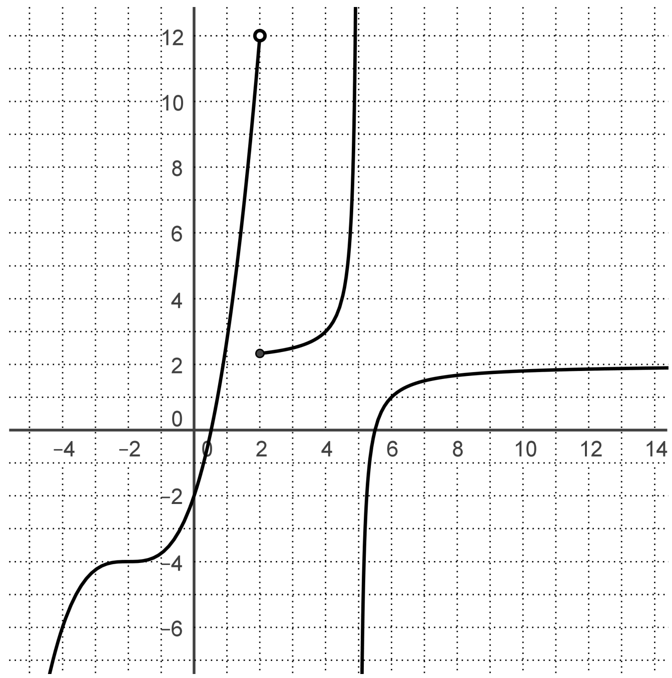
39. $\lim\limits_{x\to 5^-} h(x)$

40. $\lim\limits_{x\to 5^+} h(x)$

41. $\lim\limits_{x\to 5} h(x)$

42. $\lim\limits_{x\to \infty} h(x)$

43. $\lim\limits_{x\to -\infty} h(x)$

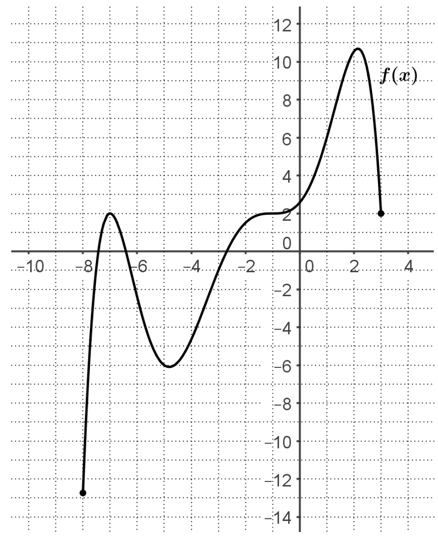


Section 2.3 Extrema and Intervals of Increase and Decrease

•Extrema• In our quest to know a function, it's reasonable to be interested in what its highest and lowest values are—either over the entire domain of a function or within a certain portion of its domain—and which input values produce these output values. The highest value a function has over its entire domain is known as its **absolute (or global) maximum value**, while the lowest value it has over its entire domain is known, as you might predict, as its **absolute (or global) minimum value**.

Think About It 2.3.1 Identify the absolute maximum and absolute minimum values of the function $f(x)$ and state the value of x where each occurs.

Think About It 2.3.2 How would your answer to TAI 2.3.1 change if the endpoints of the graph were changed to open dots?



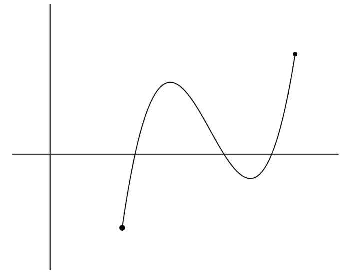
It's also useful to know where, informally speaking, the tops of hills, peaks of mountains, or bottoms of valleys in the graph of a function occur. At the top of a a hill, the function may or may not reach its absolute maximum value, but it certainly has a higher or lower value than it has as at the points immediately to either side. To speak formally of such points, we have the following definitions:

Definition 2.3.1 A function f has a **relative (or local) maximum value** of $f(c)$ at $x = c$ if and only if there exists an open interval (a, b) containing c such that $f(c) \geq f(x)$ for all x in (a, b).

Definitions are fundamental to mathematics, and reading (and writing them) is a skill and art that takes a great deal of practice, but is a hugely important piece of mathematical literacy, critical to understanding the structure of mathematics. Did you skip over that definition thinking it looks too complex to process or did you read it really carefully and critically, working hard to process the notation and language? Or something in between?

If you did not spend enough time reading and thinking about Definition 2.3.1 to fully understand it or to have written down some well-formulated questions about it, go back and read a tiny bit at a time. Try to complete TAI 2.3.3 in conjuction with your re-reading. Which pieces of the definition (if any) *do* make sense to you? For a piece that doesn't make sense to you, write down a question about what is baffling. Where there are letters in the notation, can you figure out what numbers you could (and couldn't) replace them with from a specific example? Ask yourself *why* a particular piece of the definition is included at all or is constructed the way it is. Where you can imagine something slightly different in the definition (for example, reference to a closed interval instead of an open interval)think about the consequences of making that tweak to the definition. Mathematical definitions do not lend themselves to speed reading.

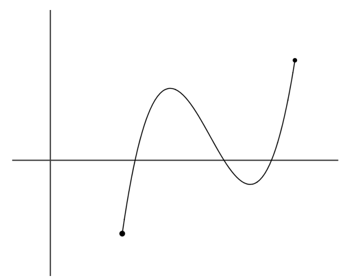
Think About It 2.3.3 Provide an illustration to accompany Definition 2.3.1 by labeling the axes in the diagram at right with a set of possible locations for a, b, c, and $f(c)$.



Think About It 2.3.4 How is the role of a definition in mathematics different than the role of a definition in other disciplines or in everyday discourse? What effect does that have on how a mathematical definition is written and used?

Definition 2.3.2 A function f has a **relative (or local) minimum value** of $f(c)$ at $x = c$ if and only if there exists an open interval (a, b) containing c such that $f(c) \leq f(x)$ for all x in (a, b).

Think About It 2.3.5 Provide an illustration to accompany Definition 2.3.2 by labeling the axes in the diagram at right with a set of possible locations for a, b, c, and $f(c)$.



Collectively, we refer to maxima and minima as **extrema**. (Maxima, minima, and extrema are the plural forms of maximum, minimum, and extremum.)

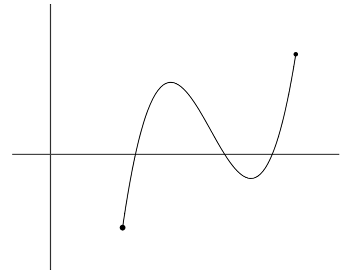
Think About It 2.3.6 What conclusions can you draw when you consider the definitions of extrema in the case where f is a constant function?

•Intervals of Increase and Decrease• Informally, another thing to think about when getting to know a function is where a bug traveling from left to right would be climbing uphill and where it would be going downhill. We say the function is **increasing** anywhere the bug would be climbing uphill and **decreasing** anywhere it would be going downhill.

Definition 2.3.3 Consider a function f which is defined everywhere on an interval I, and let a and b be values in I. If $f(b) > f(a)$ whenever $b > a$, then we say that f is **increasing** on I.

[Note that when mathematicians talk about a function being defined "on an interval I," they mean that the numbers in the interval I are *a set of possible inputs* to the function.]

Think About It 2.3.7 Provide an illustration to accompany Definition 2.3.3 by labeling the axes in the diagram at right with a set of possible locations for p, q, a, b, $f(a)$, and $f(b)$.



Think About It 2.3.8 Using Definition 2.3.3 as a model, define a decreasing function in the box provided for Definition 2.3.4.

Definition 2.3.4 Consider a function f

Problem Set 2.3

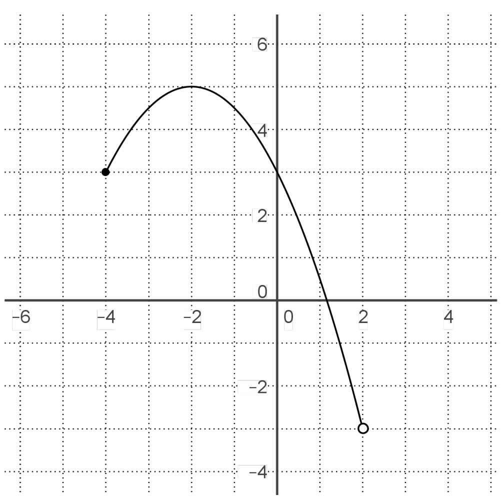
In problems 1 to 5 the graph of a function f is given. Find the following, estimating where necessary.

a) the interval(s) on which f is increasing

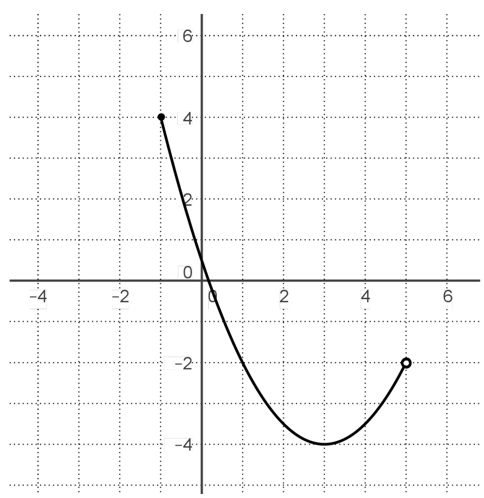
b) the interval(s) on which f is decreasing

c) the value(s) of x for which f has relative and/or global extrema, and the type of extremum and value of the function at each of these x-values

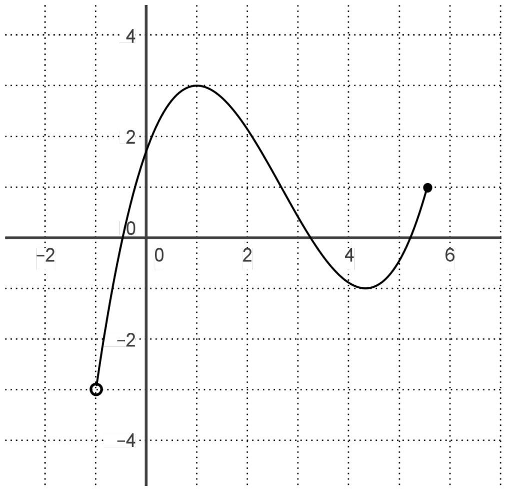
1.



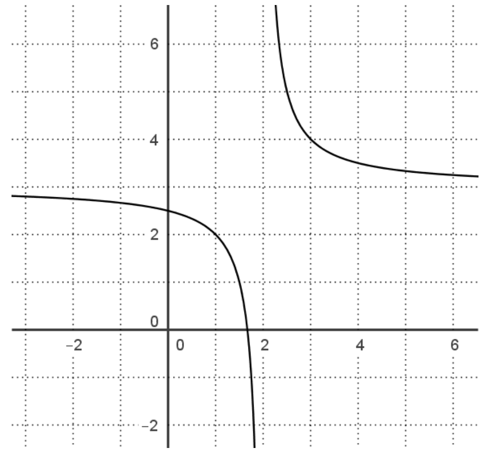
2.



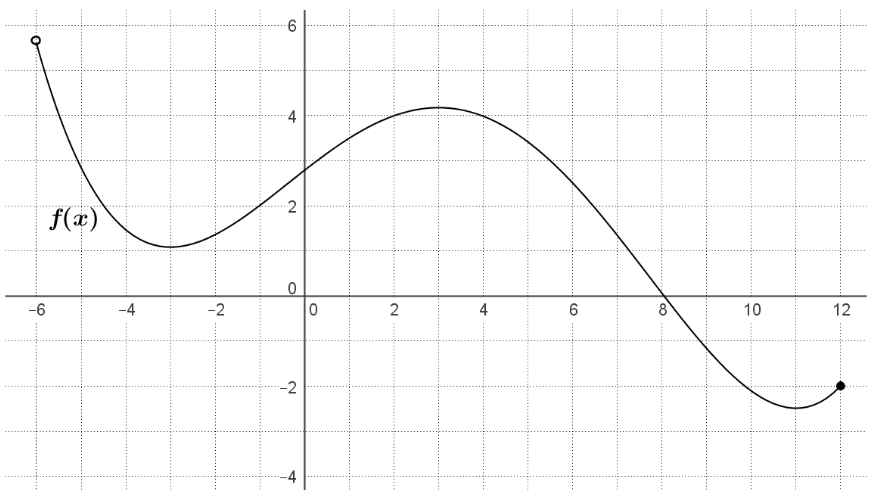
3.



4.



5.



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6. A projectile is launched straight up into the air in such a way that $h(t)$, its height as a function of the time since it was launched, is given by the quadratic graph shown. Find each of the following *and explain what specific information about the situation each provides.*

a) domain of h

b) range of h

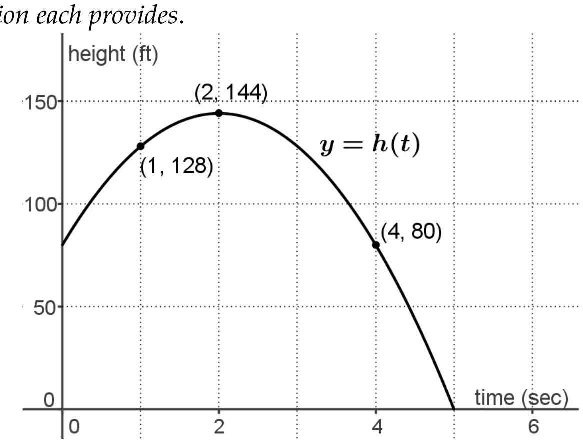
c) intercepts of h

d) interval(s) on which h is increasing

e) value(s) of t for which $h(t) \geq 128$

f) $h(2) - h(0)$ (include units in your answer)

g) $\dfrac{h(b) - h(a)}{b - a}$ if $a = 1$ sec and $b = 4$ sec (include units in your answer)



7. Consider the function $f(x)$ which is increasing on its entire domain of $(-10, 10]$. Classify each statement as true or false and explain how you know.

a) It is impossible for the function to have any local extrema.
b) The function has a global minimum value at $x = 10$.
c) The function has an absolute maximum value of $f(10)$

8. Consider the function $g(x)$ with domain $[0, 8]$ and range $[-3, 12]$. If $g(x)$ is decreasing on $[0, 5]$ and increasing on $[5, 8]$, identify all possible extreme values of g as well as their type and the values of x where they could occur. Sketch graphs to support your answer.

In problems 9 to 14, use graphing technology to help you identify all values of x for which the given function has local or absolute extrema. For each value of x you give, classify the extremum as a local maximum, a local minimum, an absolute maximum, or an absolute minimum and tell what the value of the function is at that x-value.

9. $f(x) = \dfrac{x+3}{x^2 - 7x + 10}$

10. $f(x) = x^2 (x-6)^2 - 10$

11. $f(x) = x^3 + \frac{3}{2}x^2 - 6x$

12. $f(x) = (x+4)\,|x-2|$

13. $f(x) = x^2\sqrt{x+2} - 3$

14. $f(x) = \dfrac{x^4 + x^3 - 7x^2 - x + 6}{x^2 + 1}$

Section 2.4 Even and Odd Functions

Definition 2.4.1 A function f is **even** if and only if $f(-x) = f(x)$

Definition 2.4.2 A function f is **odd** if and only if $f(-x) = -f(x)$

Think About It 2.4.1 Make up any two functions. For each function find the output for the inputs of 1, 2, 3, −1, −2, and −3. What do the results suggests about whether the functions you've made up are even, odd, or neither?

Think About It 2.4.2 Consider a function whose graph includes the points $(-4, 1)$, $(-2, -3)$, $(6, -9)$, and $(10, 15)$. What other points must be on the function's graph if the function is even? odd?

Think About It 2.4.3 What can you say about the intercepts of even and odd functions?

Think About It 2.4.4 What do the graphs of all even functions have in common? Odd functions?

Problem Set 2.4

1. If $g(-3) = 7$, find the coordinates of a point other than $(-3, 7)$ on the graph of $y = g(x)$ if
   a) g is even
   b) g is odd

2. If $g(-2) = -5$, find the coordinates of a point other than $(-2, -5)$ on the graph of $y = g(x)$ if
   a) g is even
   b) g is odd

3. State the definition of an odd function and show how to use this definition in an algebra-based proof of the fact that the function $f(x) = 2x^3 - 7x$ is odd.

4. State the definition of an even function and show how to use this definition in an algebra-based proof of the fact that the function $f(x) = -\dfrac{3x^2}{x^4 + 2}$ is even.

5. State the definitions of even and odd functions and show how to use these definitions in an algebra-based proof of the fact that the function $f(x) = 3x^3 + 5x - 1$ is neither even nor odd.

6. Determine whether $f(x) = |x^3 - 4x|$ is even, odd, or neither. Show how to use the definitions of even and/or odd functions to prove you are correct.

Section 2.5 Power Functions and their Transformations

Definition 2.5.1 A power function is a function of the form $f(x) = x^a$, where a is any real number.

Think About It 2.5.1 Find $f(8)$, $f(0)$, and $f(-8)$ for the power function $f(x) = x^a$ in each of the following cases: $a = 1$, $a = 2$, $a = -1$, $a = -2$, $a = \frac{1}{2}$, $a = \frac{1}{3}$, $a = -\frac{2}{3}$, $a = 0$

The question of how to define 0^0 is an interesting one, and good arguments can be made both for defining it to be 1 and for saying that is undefined. While you might initially also want to make an argument for defining it as equal 0, it turns out that this is rarely a useful definition, and so mathematicians do not argue for it. (Try typing 0^0 into Desmos and into WolframAlpha to see how the programmers of each of these applications have opted to define it.) There are many contexts–especially those in which we are dealing with functions whose domains are a subset of the real numbers–in which the advantages of defining 0^0 as equal to 1, far outweigh the disadvantages, and therefore it is the definition we will use. It is quite helpful to have it defined this way when we're considering the power function $f(x) = x^0$, and–according to an "Ask Dr. Math" post at the well-respected mathforum.org–Euler argued that 0^0 should be defined as equal to 1 for this reason. As you sketch the graphs and think about the domains of power functions, think about why this is.[3]

[3]http://mathforum.org/dr.math/faq/faq.0.to.0.power.html. Accessed July 31, 2016.

Think About It 2.5.2 What does it mean to you about math that sometimes we make definitions (or leave things undefined) just so that things are more convenient?

Think About It 2.5.3 Determine whether its possible to have a power function that has the given domain or range. If it is possible, give a couple of examples. If not, explain how you know.

a) domain of $(-\infty, \infty)$

b) range of $(-\infty, \infty)$

c) domain of $[0, \infty)$

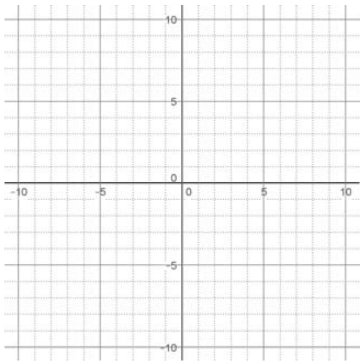
d) domain of $(-\infty, 0]$

e) range of $[0, \infty)$

f) domain of $(-\infty, 0) \cup (0, \infty)$

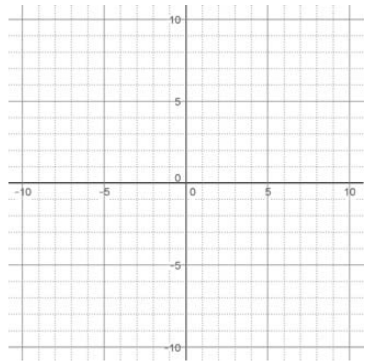
Absolutely Vital Functions 1: Power Functions

$f(x) = x^0$ (2 pts. required)



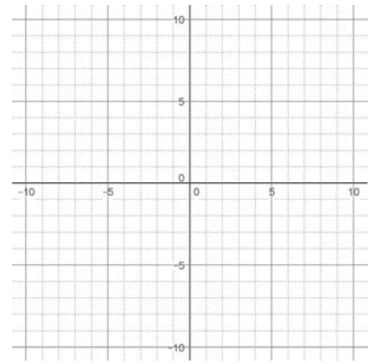
domain: range:

$f(x) = x$ (2 pts. required)



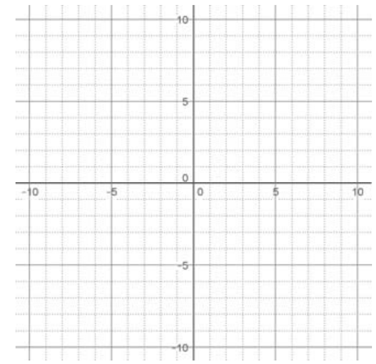
domain: range:

$f(x) = x^2$ (7 pts. required)



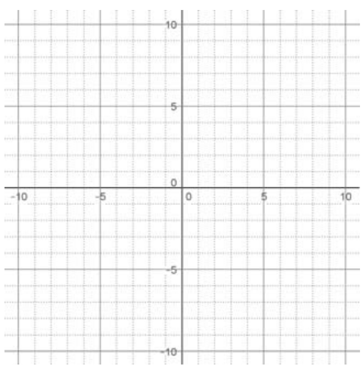
domain: range:

$f(x) = x^3$ (5 pts. required)



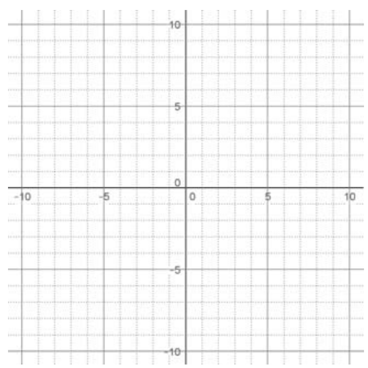
domain: range:

$f(x) = x^{1/2}$ (4 pts. required)



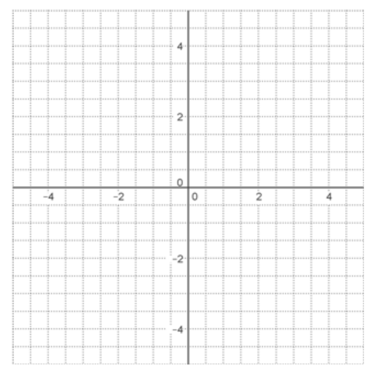
domain: range:

$f(x) = x^{1/3}$ (5 pts. required)



domain: range:

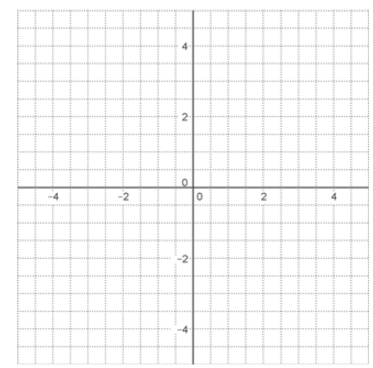
$f(x) = x^{-1}$ (6 pts. required)



domain: range:

equations of asymptotes:

$f(x) = x^{-2}$ (6 pts. required)



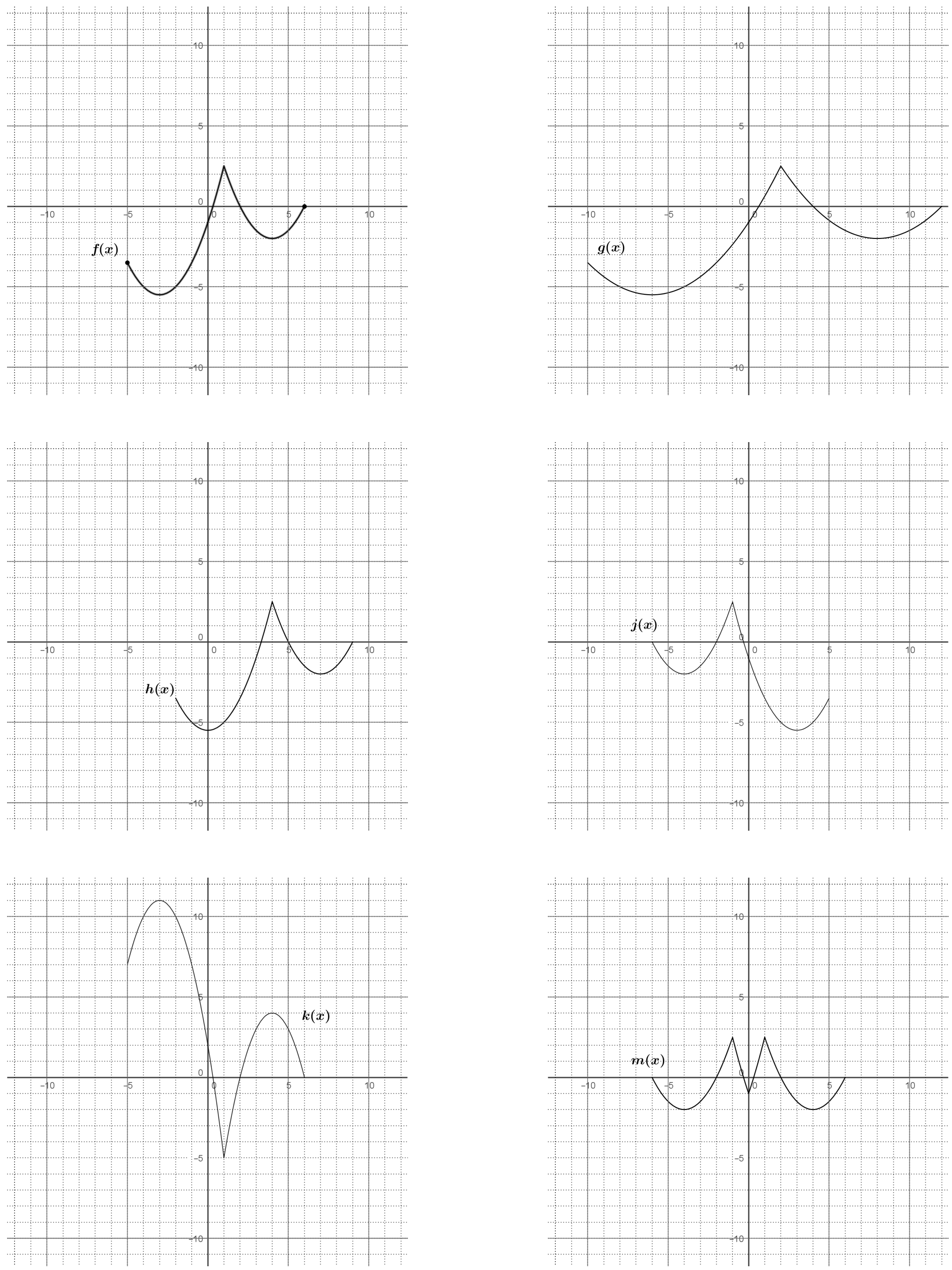
domain: range:

equations of asymptotes:

Once we have developed a decent acquaintance with power functions, we can vastly expand our knowledge of functions in general by exploring what happens when we create new functions (which we might think of as children) by defining variations on these old functions (which we can think of as the parents) that we've come to know well.

Exploration 2.1

For each function shown, write down the coordinates of the endpoints and relative extrema. Then explain in words what transformation(s) you could apply to the graph of f to obtain the graph of each of the other functions. Finally, think about and what adjustments you would need to make to the equation $y = f(x)$ to produce these transformations.



Think About It 2.5.4 Classify a variety of power functions and some transformations of them as even, odd, or neither.

Exploration 2.2

Imagine applying each of the transformations you know about to an even function. For each, determine whether the transformed function will remain even sometimes, always, or never. What about if the original function is odd?

Problem Set 2.5

In problems 1 to 21,

a) Tell whether the function is a power function or a transformation of a power function.

b) Carefully plot the number of points specified and use these points to help you make a nice graph of the function (on graph paper!).

c) Use the graph to determine the domain and range of the function.

d) Explain why the domain and range seem reasonable when you think about the equation.

e) If the function is a transformation of a power function, describe how its graph is related to the graph of the power function that it is a transformation of.

f) Tell whether the function is even, odd, or neither.

1. $f(x) = 1$ {use 2 points}

2. $f(x) = 3$ {use 2 points}

3. $f(x) = -\frac{5}{2}$ {use 2 points}

4. $f(x) = x$ {use 2 points}

5. $f(x) = -\frac{1}{2}(x-3) - 2$ {use 2 points}

6. $f(x) = x^2$ {use 5 points}

7. $f(x) = \dfrac{x^2}{4}$ {use 5 points}

8. $f(x) = \left(\dfrac{x}{3}\right)^2$ {use 5 points}

9. $f(x) = -x^2 + 1$ {use 5 points}

10. $f(x) = x^3$ {use 5 points}

11. $f(x) = -2x^3$ {use 5 points}

12. $f(x) = x^{1/2}$ {use 4 points}

13. $f(x) = (-x)^{1/2}$ {use 4 points}

14. $f(x) = x^{1/3}$ {use 5 points}

15. $f(x) = (x+1)^{1/3}$ {use 5 points}

16. $f(x) = x^{-1}$ {use 6 points}

17. $f(x) = x^{-1} - 2$ {use 6 points}

18. $f(x) = x^{-2}$ {use 6 points}

19. $f(x) = (x+2)^{-2}$ {use 6 points}

20. $f(x) = |x|$ {use 3 points}

21. $f(x) = |x| - 3$ {use 3 points}

In problems 22 to 27, a function f is given and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph. (Your answers should not include the letter f.)

22. $f(x) = \dfrac{1}{x}$ is shifted 4 to the right and 7 down

23. $f(x) = \sqrt{x}$ is reflected over the y-axis and stretched vertically by a factor of 3

24. $f(x) = \sqrt[4]{x}$ is reflected over the x-axis and compressed horizontally by a factor of 5

25. $f(x) = \dfrac{1}{x}$ is shifted 2 to the left and 6 down

26. $f(x) = \dfrac{1}{x}$ is stretched vertically by a factor of 2 and moved 3 to the left

27. $f(x) = \sqrt{x}$ is reflected over the x-axis and stretched horizontally by a factor of 5

In problems 28 to 31 make a careful graph (on graph paper) of the function.

28.
$$f(x) = \begin{cases} -x & , \quad x \le -3 \\ x^2 - 6 & , \quad -3 < x < 1 \\ x^{1/2} & , \quad x \ge 1 \end{cases}$$

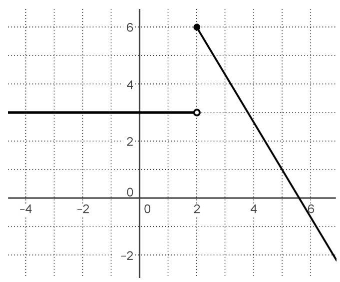
29.
$$f(x) = \begin{cases} -3 & , \quad x \le -6 \\ 3(x+5) & , \quad -6 < x < -3 \\ x^{-1} & , \quad x \ge -3 \end{cases}$$

30.
$$f(x) = \begin{cases} x^{-1} & , \quad x < 2 \\ x+1 & , \quad 2 \le x \le 5 \\ -(x-5)+6 & , \quad x > 5 \end{cases}$$

31.
$$f(x) = \begin{cases} 2(x+4)+7 & , \quad x < -5 \\ |x| & , \quad -5 < x < 1 \\ -x^{1/2} & , \quad x > 1 \end{cases}$$

In problems 32 to 35 write an equation for the function and state its domain and range.

32.



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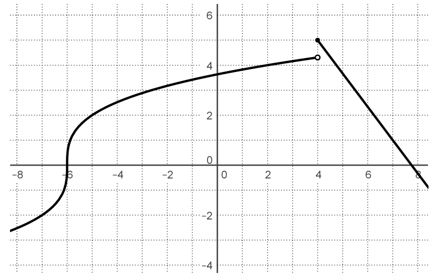

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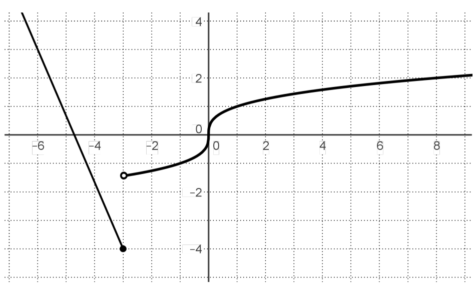

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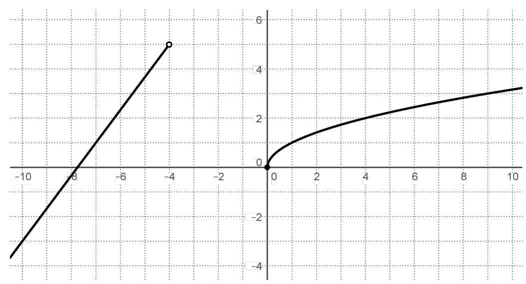

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36. Use the given graph of h to graph f.

a) $f(x) = 2h(x)$

b) $f(x) = h(2x)$

c) $f(x) = h\left(\frac{x}{2}\right)$

d) $f(x) = \frac{3h(x)}{2}$

e) $f(x) = h(0.1x)$

f) $f(x) = -2h(x)$

g) $f(x) = h(-2x)$

h) $f(x) = h(x-5)+2$

i) $f(x) = h(x-3)-4$

j) $f(x) = 3 - h(x)$

k) $f(x) = -4 - h(2x)$

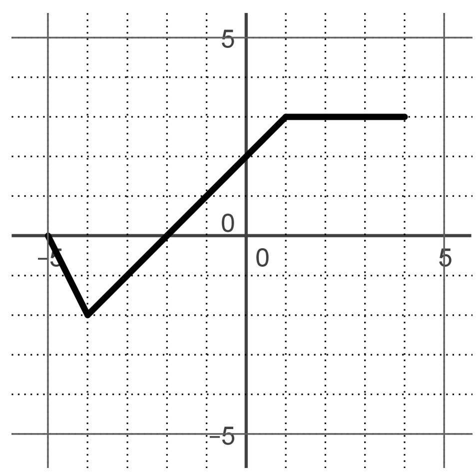
l) $f(x) = |h(x)|$

m) $f(x) = |h(x-2)|$

n) $f(x) = h(|x|)$

o) $f(x) = -h(|x|)$

p) $f(x) = |h(-x)|$



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37. Explain how the graph of $g(x) = 3-(x+5)^2$ can be obtained by stretching, shrinking, shifting and/or flipping the graph of $f(x) = x^2$. Then graph $y = g(x)$ on graph paper. Be sure to plot at least 5 points with care.

38. Explain how the graph of $g(x) = 3\sqrt{x+2} - 4$ can be obtained by stretching, shrinking, shifting and/or flipping the graph of $f(x) = \sqrt{x}$. Then graph $y = g(x)$ on graph paper. Be sure to plot at least 4 points with care.

39. Explain how the graph of $g(x) = -3 + 2\sqrt{-x}$ can be obtained by stretching, shrinking, shifting and/or flipping the graph of $f(x) = \sqrt{x}$. Then graph $y = g(x)$ on graph paper. Be sure to plot at least 4 points with care.

40. Consider the function $f(x) = (x-8)^{-2} - 3$.

a) Give the equations of any asymptotes of f.
b) State the domain and range of f.
c) Write limit expressions to describe the behavior of f near its asymptotes

41. If you know that $p(12) = -5$, identify one point on the graph of

a) $y = 2p(x)$ b) $y = p(-x)$ c) $y = p(x+3)$ d) $y = p\left(\dfrac{x}{3}\right)$

42. If you know that $p(-4) = 10$, identify one point on the graph of

a) $y = p(2x)$ b) $y = -p(x)$ c) $y = p(x-1)+5$ d) $y = 2p(-x)$

43. If you know that $p(-6) = 8$, identify one point on the graph of

a) $y = p(-x)$ b) $y = p(x-3)+4$ c) $y = p(10x)$ d) $y = 2-3p(x)$

44. The function $f(x) = 64 - 10(x+42)^2$ has either a local maximum or a local minimum. State which type of extremum it has, tell what this local maximum or local minimum value is, and state the value of x where it occurs.

45. The function $f(x) = -110 + 3(x-54)^2$ has either a local maximum or a local minimum. State which type of extremum it has, tell what this local maximum or local minimum value is and state the value of x where it occurs.

46. The GS varsity Scrabble team holds a rally at 5:00 pm. At 5:06, 61 Cougar Crazies are there, and at 5:42, 133 Cougar Crazies are there. The rally ends at 6:30 pm.

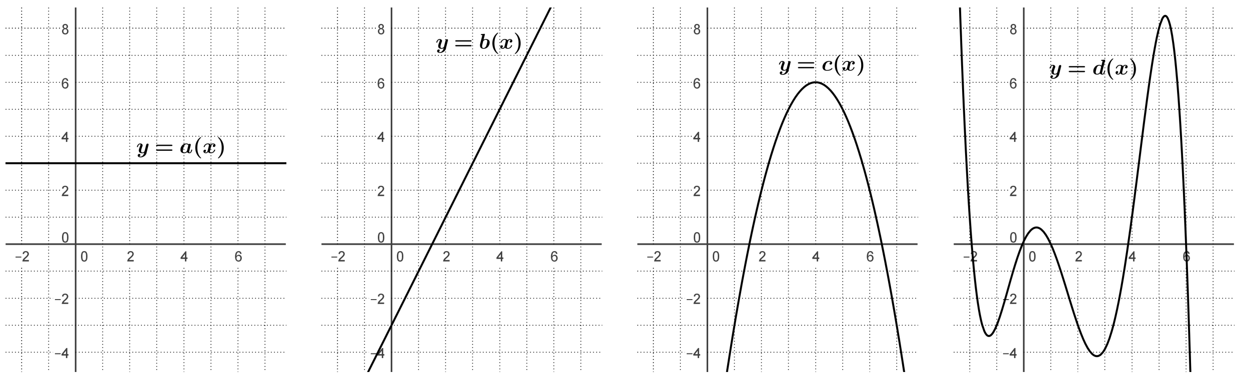
a) Assuming the number of Cougar Crazies at the rally can be modeled by a linear fuction, write an equation for $P(t)$, the number of Cougar Crazies at the rally as a function of the number of minutes that have passed since 5:00 pm.
b) State the domain and range of P.

47. Reflect some more on the question of what the value of 0^0 is, should be, or might be. How do the choices we make as mathematicians decide the quality and amount of math we are able to describe or understand?

Section 2.6 Average Rate of Change

The average rate of change of a function from one input value to another tells us how much the value of the function would change for each one unit change in the input if it always went up by the same amount. So, for example, if we know that a population changed by 100,000 people over the course of 20 years, we would say that the average rate of change of the population during that time was 5,000 people per year, though its entirely possible that the population changed by a different amount every year.

Think About It 2.6.1 Which of the functions shown below have the same average rate of change on the interval $[1, 5]$?



Think About It 2.6.2 What is the average rate of change of the function $Q(x) = \frac{x^2}{2}$ on the interval $6 \le x \le 10$?

Think About It 2.6.3 Write an algebraic expression involving f, a, and h which represents the average rate of change for any function $f(x)$ from the point where the input is some value a to the point where the input is h units more than a.

Think About It 2.6.4 Write down a function which has a negative rate of change on any interval $[a, b]$ in the function's domain. (Assume $b > a$.)

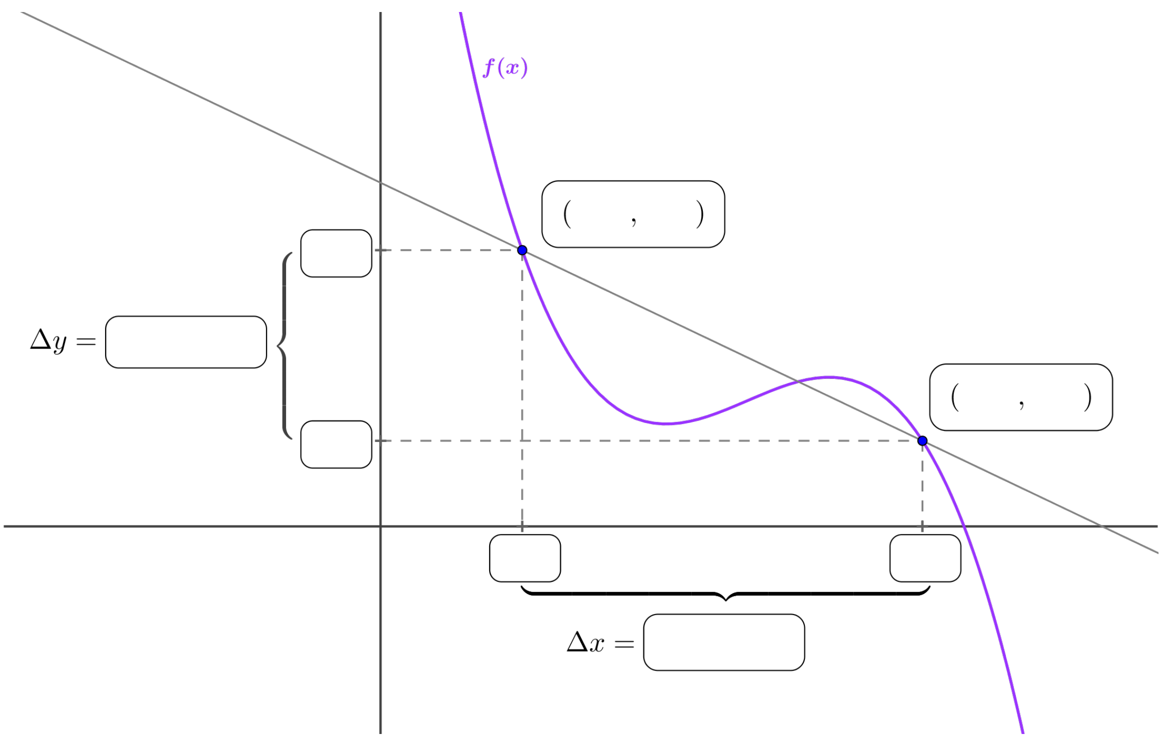
Think About It 2.6.5 Is the average rate of change for the function you wrote down in TAI 2.6.4 the same for any choice of a and b? If so, find a function for which the average rate of change is always negative, but changes depending on the values of a and b.

A line passing through any two points on a curve is known as a **secant** line. The problem of finding the average rate of change of a function over an interval is identical to finding the slope of the secant line through the endpoints of that interval. (Can you convince yourself of this?) Knowing this may help you make sense of the formal definition of average rate of change.

Definition 2.6.1 The **average rate of change** of a function f on the interval $[a, a+h]$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a}$$

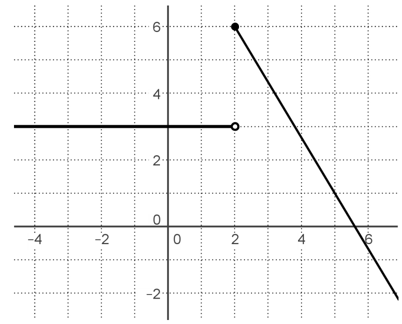
Think About It 2.6.6 Fill in the boxes in the diagram to illustrate Definition 2.6.1.



Think About It 2.6.7 How could the expression for the average rate of change in Definition 2.6.1 be simplified? Why do you think the writer of this definition chose not to write the expression in simplified form?

Problem Set 2.6

1. Find the average rate of change of the function on the specified interval. Assume that the function has a domain of all reals.



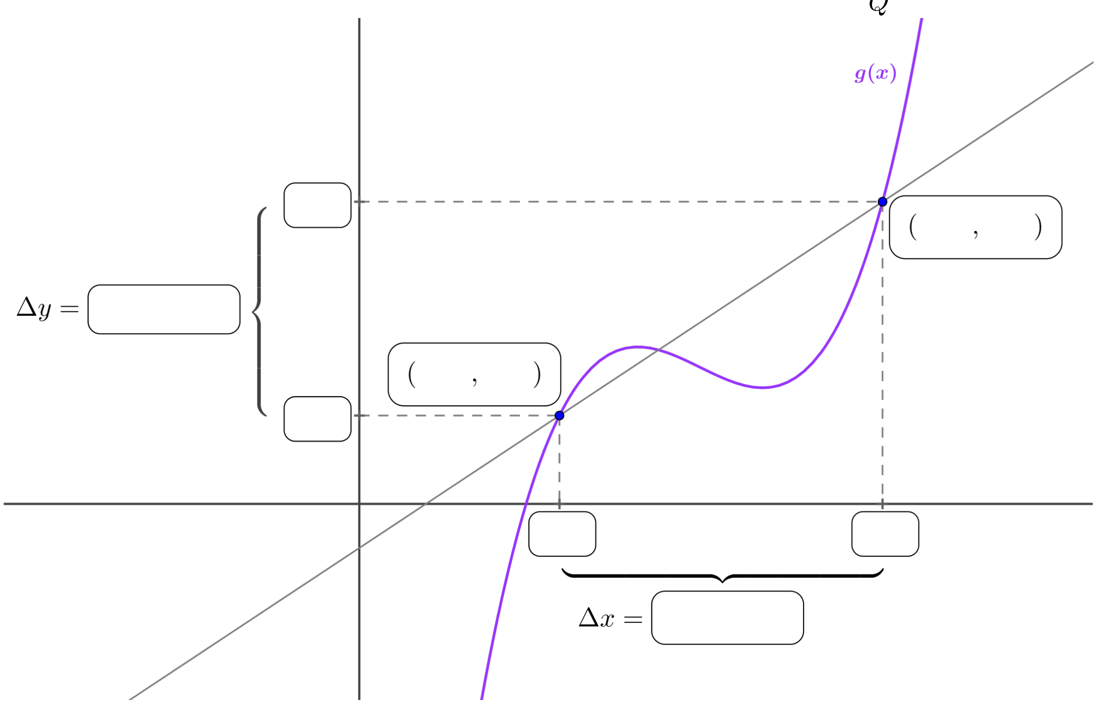
a) $[2, 5]$
b) $[6, 100]$
c) $[-2, 1]$
d) $[-50, -8]$
e) $[-2, 5]$
f) $[0, 2]$

2. A pot of boiling water is removed from the stove and sits on the counter in a warm kitchen. The temperature of the water (in degrees Celsius) is a function of the time (in minutes) since the pot was removed from the stove. Sketch a possible graph of the function, including some numbers that seem reasonable on the axes. After sketching your graph, consider the following questions and adjust your graph if necessary.

a) Should 0 be included in the domain of this function? Why or why not?
b) Is there an upper limit on the domain? Why or why not?
c) Should the graph of this function have an x-intercept? If so, what information does this intercept provide? If not, why not?
d) What is the range of this function?
e) When is the average rate of change of the function positive? negative? zero?
f) How does the absolute value of the average rate of change of the temperature of the water in the first 30 minutes compare to the absolute value of the average rate of change in the next 30 minutes and what does this tell you about the water

3. Explain, with as much specificity as you can muster, what the difference quotient $\dfrac{g(c+k)-g(c)}{k}$ tells you about the graph of g. (And why is it called a difference quotient?)

4. Fill in the boxes in the diagram to illustrate the difference quotient $\dfrac{g(p+Q)-g(p)}{Q}$.



5. What can you say about the value of the difference quotient in the case where g is a constant function? Justify your answer.

6. What can you say about the value of the difference quotient in the case where g is a linear function? Justify your answer.

7. Write an expression which represents the average rate of change of a function $P(t)$ from $t = c$ to $t = d$

In problems 8 to 23, simplify the expression $\dfrac{f(a+h) - f(a)}{h}$ for the given function.

8. $f(x) = 3$

9. $f(x) = -7$

10. $f(x) = x$

11. $f(x) = -4x + 5$

12. $f(x) = \frac{2}{3}x - 6$

13. $f(x) = 5 - \frac{1}{2}x$

14. $f(x) = 3x - x^2$

15. $f(x) = 1 - 10x^2$

16. $f(x) = x^2 - 5x + 1$

17. $f(x) = 2x^2 - 7x$

18. $f(x) = x^2 - 4x + 5$

19. $f(x) = 3x^2 - 2x + 1$

20. $f(x) = px^2 + qx + r$

21. $f(x) = x^{-1}$

22. $f(x) = x^{-2}$

23. $f(x) = x + x^{-1}$

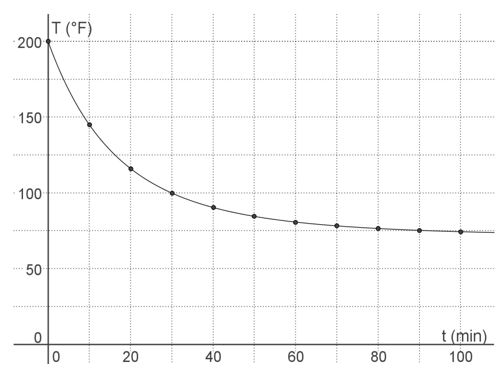
---

24. Simplify the expression $\dfrac{f(1+h) - f(1)}{h}$ in the case where $f(x) = 2x^2 - x + 4$. Then describe very specifically what information this expression provides about the graph of f in the case where $h = 6$?

25. Simplify the expression $\dfrac{g(2+c) - g(2)}{c}$ if $g(x) = \dfrac{3}{5x}$. Then describe very specifically what information this expression provides about the graph of g in the case where $c = 9$.

26. Simplify the expression $\dfrac{g(-4+a) - g(-4)}{a}$ if $g(x) = \dfrac{3}{2x+1}$. Then describe very specifically what information this expression provides about the graph of g in the case where $a = -1$.

27. The temperature, T (in °F), of a cup of tea sitting on a table t minutes after it is put on the table is shown in the graph. Find the average rate of change in the temperature of the tea from $t = 30$ min to $t = 90$ min. Remember to include units in your answer.



28. Find the average rate of change of $f(x) = \dfrac{2}{x+3}$ from $x = 1$ to $x = a$.

29. Find the average rate of change of $f(x) = \dfrac{3}{x-1}$ from $x = 5$ to $x = a$.

30. Find the average rate of change of $f(x) = \dfrac{4}{x-2}$ from $x = 3$ to $x = a$.

31. Write a fully simplified expression which gives the average rate of change of $R(x) = x^2 - 8$ from $x = a$ to $x = b$. Then draw a carefully labeled graph that illustrates a choice of values for a and and b for which the average rate of change of R on $[a, b]$ will be 2.

32. Write a fully simplified expression which gives the average rate of change of $R(x) = 2 - \frac{1}{2}x^2$ from $x = a$ to $x = b$. Then draw a carefully labeled graph that illustrates a choice of values for a and and b for which the average rate of change of R on $[a, b]$ will be 3.

Section 2.7 Inverses of Functions

Definition 2.7.1 If each value in the range of a function corresponds to exactly one value in the domain, then the function is said to be **one-to-one**.

Think About It 2.7.1 Use the definition of one-to-one functions to determine which of the following functions are one-to-one and which are not: $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^{-1}$, and $f(x) = x^{1/2}$. For those that are not one-to-one, give some examples of multiple inputs producing the same output.

Think About It 2.7.2 How can you tell from the graph of a function whether or not it is one-to-one?

Think About It 2.7.3 Are even functions sometimes, always, or never one-to-one? Odd functions? How do you know?

Functions that undo the work of each other are known as inverse functions. In more formal mathematical language this is expressed in the following definition:

Definition 2.7.2 $f(x)$ and $g(x)$ are said to be **inverse functions** if and only if f and g are one-to-one functions with the property that $f(g(x)) = g(f(x)) = x$.

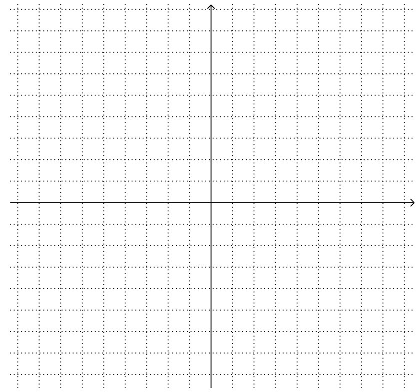
The notation $f^{-1}(x)$ is used to denote the inverse of f. It is crucial to note that this ***does NOT*** mean $\dfrac{1}{f(x)}$!

Think About It 2.7.4 How does the formal definition of inverse functions show that f and g "undo" the effects of each other?

Think About It 2.7.5 Find $f^{-1}(x)$ for the functions you identified as one-to-one in TAI 2.7.1.

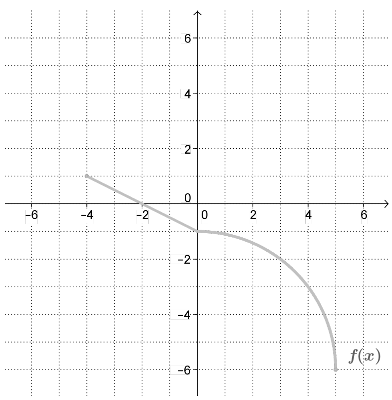
Think About It 2.7.6 Create a function $f(x)$ by choosing a one-to-one power function and applying at least two transformations. Then find $f^{-1}(x)$ and show that $f(f^{-1}(x)) = x$ and that $f^{-1}(f(x)) = x$.

Think About It 2.7.7 What is the relationship between the graph of a function and the graph of its inverse?

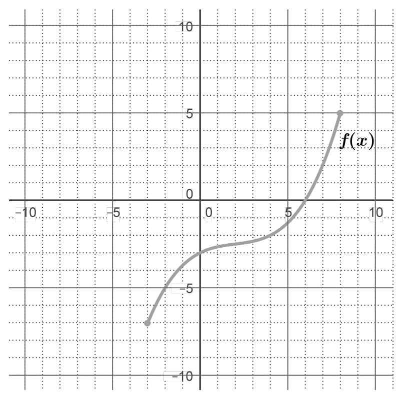


Problem Set 2.7

1. If $f(x)$ is the function graphed below, graph $y = f^{-1}(x)$, plotting at least five points with care.



2. If $f(x)$ is the function graphed below, graph $y = f^{-1}(x)$, plotting at least five points with care.



3. If $g(-3) = 7$, find the coordinates of a point on the graph of $y = g^{-1}(x)$ if g is invertible

4. If $g(-2) = -5$, find the coordinates of a point on the graph of $y = g^{-1}(x)$ if g is invertible

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In problems 5 to 13, find $f^{-1}(x)$.

5. $f(x) = \sqrt[3]{2x-5}$

6. $f(x) = 4 - \sqrt[5]{7x}$

7. $f(x) = \dfrac{6}{3x+2}$

8. $f(x) = \dfrac{2}{x^3-5}$

9. $f(x) = \dfrac{6}{2x^{3/5}+1}$

10. $f(x) = 3 + \sqrt{x+5}$

11. $f(x) = 1 - \sqrt{x+4}$

12. $f(x) = \dfrac{x}{x+4}$

13. $f(x) = \dfrac{x+5}{2x-3}$

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14. Find $\left(f \circ f^{-1}\right)(367)$ if $f(x) = 2x - 6$

15. If $g(x) = \sqrt[3]{x+3}$, find a function $w(x)$ such that $(g \circ w)(x) = x$

16. Consider the function $B(t) = (t-7)^2 + 4$, $t \leq a$. For what values of a will B^{-1} exist? Explain.

Section 2.8 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

Problem Set 2.8

1. The function $f(x)$ is graphed at right. Find the following, estimating where necessary.

a) the domain and range of f

b) the interval(s) on which f is increasing

c) the value(s) of x for which f has relative and/or global extrema, and the type of extremum and value of the function at each of these x-values

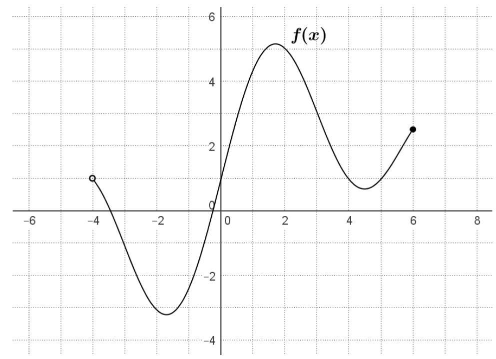
d) the solution(s) to the equation $f(x) = 0$

e) the solution(s) to the equation $f(x) = -\frac{2}{5}(x+2)+5$

f) the solution to the inequality $f(x) \le 1$

g) $(f \circ f)(2)$

h) the domain of $h(x) = \sqrt{-f(x)}$



2. Consider the functions $Q(x) = 3 - x^3$ and $R(x) = \sqrt{1-2x}$

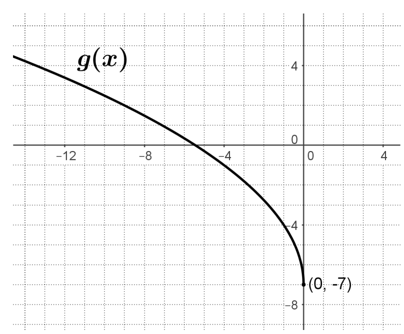
a) State the domains of Q and R.
b) Find $(R \circ Q)(x)$ and simplify.
c) Write the limit expressions that describe the end behavior of Q and R.

3. Classify each function as even, odd, or neither

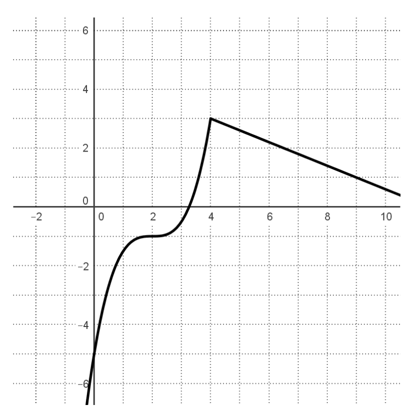
a) $f(x) = 9$ b) $f(x) = x^{4/5}$ c) $f(x) = -2\sqrt[3]{x}$ d) $f(x) = (x-4)^2$

4. If $q(-6) = 20$ and $q(10) = -8$, find the coordinates of **two points** other than $(-6, 20)$ and $(10, -8)$ that would be on the graph of

a) $y = -4 + q(2x)$ b) $y = q(x)$ if q is odd c) $y = q(x)$ if q is even



5. Consider the function g graphed at right. Assuming that g has a domain of $(-\infty, 0]$ and a range of $[-7, \infty)$ and is a transformation of a power function, find its equation.



6. The graph of the function $P(x)$, with domain $(-\infty, \infty)$, is shown. Write a piecewise equation for P if the pieces of P are transformations of power functions.

7. Use the graph of P from problem 6 to find the average rate of change of P on

a) $[2, 9]$ b) $[213, 777]$

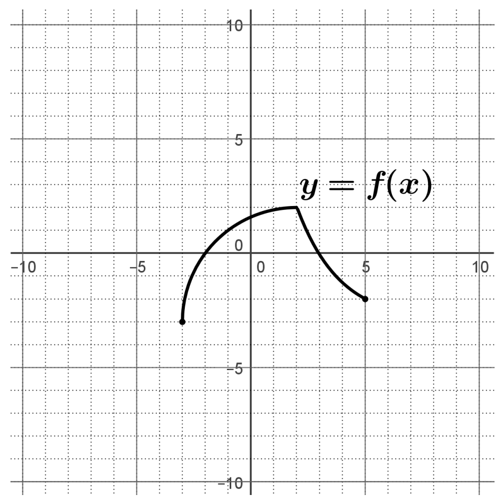
8. Consider the function $P(x) = 25 - \dfrac{10}{x+7}$

a) Identify the power function that P is a transformation of and state, in order, the transformations that must be applied to this power function to produce P. Be specific.

b) Give the equations of the asymptotes of P and write the limit expressions that describe the behavior of P as it approaches its asymptotes.

c) Find $P^{-1}(x)$

In problems 9 to 10, use the given graph of $f(x)$ to answer the questions.



9. Graph $g(x) = 2f(|x|)$ and $h(x) = |f(x+2)|$.

10. Let $g(x) = -2f(x+1) + 3$.

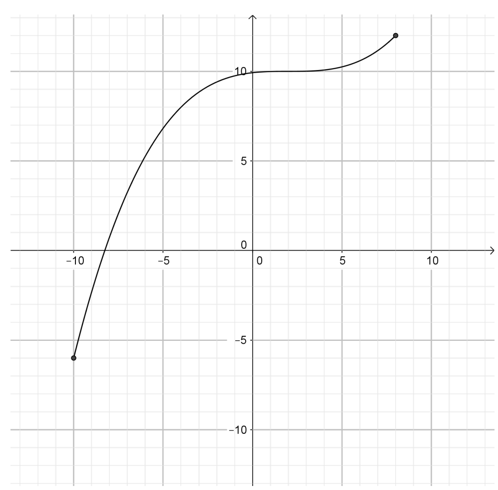
a) Without first making a graph of $g(x)$, write down what you expect its domain and range to be.

b) Describe the steps required to find the point on g that corresponds to a given point on f.

c) Find the point on g that corresponds to each of the following points on f: $(-3, -3)$, $(2, 2)$, $(5, -2)$

d) Choose any two other points on f and find the corresponding points on g.

e) Graph g and compare the domain and range of your graph to your prediction in part (a).



11. The graph of f is shown at right.

a) Graph f^{-1}.

b) State the domain and range of f^{-1}.

c) Describe how you could have determined the domain and range of f^{-1} without graphing it.

d) State the domain and range of $3f(2x)$.

e) State the domain and range of $f(|x|)$.

f) State the solution to $|f(x)| \le 2$.

12. Simplify the expression $\dfrac{Q(w+r) - Q(w)}{r}$ if $Q(x) = x^2 - 3x + 4$.
Then describe very specifically what information this expression provides about the function Q in the case where $w = 2$ and $r = 7$.

13. Tell whether the statement is sometimes, always, or never true. Explain your choice.

a) If $h(x) = f(|x|)$, then h is an even function.

b) If $h(x) = |f(x)|$ then the range of h is $[0, \infty)$.

Exploration 2.3

Try to get to know some or all of the following functions using the various lenses you've been considering in this chapter. What can you discover about them?

- the function whose output is the largest integer less than or equal to the input
- the function whose output is the distance between 3 and the input on the number line
- the function whose output is the exponent that you must put on 2 to get the input
- the function whose input is the distance that a bug starting at the point $(1, 0)$ and moving counter-clockwise has traveled on the circle $x^2 + y^2 = 1$ and whose output is the bug's x coordinate
- the function whose input is a positive integer and whose output is the sum of all of the factors of the input

3 Further Function Families

Section 3.1 Exponential and Logarithmic Functions

Think About It 3.1.1 Describe a real-life situation that could be modeled by each of the following equations. Tell what the variables and constants would represent. Sketch a graph of each function, labeling some key values on each axis to indicate scale.

- $f(t) = 1000 + 2t$

- $g(t) = 1000 \cdot 2^t$

- $h(t) = 1000 \cdot (1.12)^t$

Hint: If you're having trouble getting started, one possible approach is to input $t = 0$, $t = 1$, $t = 2$, and $t = 3$ and look for a pattern in the function's outputs. (You may also want to input $t = -1$ and $t = -2$ to get a better sense of the overall shape of the graph. What would these negative inputs correspond to in your real-life situation?)

Definition 3.1.1 An exponential function is a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

Think About It 3.1.2 If you had to name one thing that gets to the heart of the difference between power functions and exponential functions, what would you say?

Exploration 3.1

Experiment with several different values of b in Definition 3.1.1 with the goal of identifying some characteristics that all exponential functions share and some characteristics that differ depending on the value of b. Be sure to think about the various properties of functions explored in Chapter 2. Summarize your findings here.

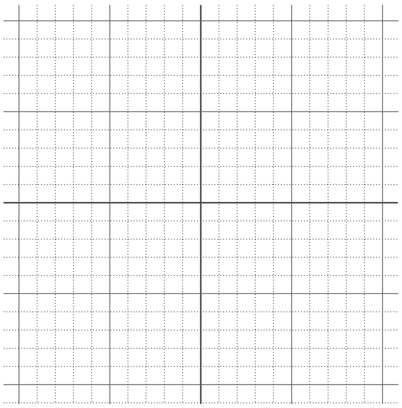
Exploration 3.2

Why are the values of b restricted the way they are in Definition 3.1.1? What happens to a function of the form given in the definition if b is allowed to take on values that the definition forbids?

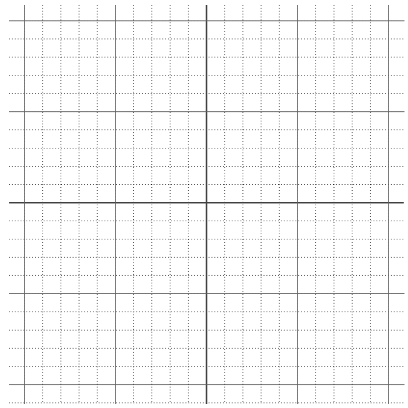
Exploration 3.3

Graph the following pairs of equations. You should see a clear relationship between the graphs in each pair. Can you give an algebraic explanation for the relationships you observe?

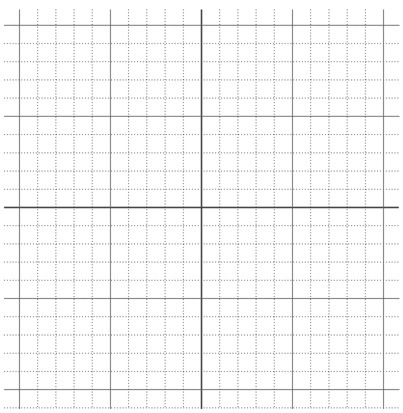
$f_1(x) = 2^{x+3}$



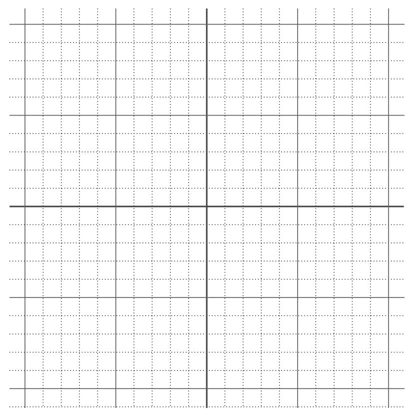
$g_1(x) = 8 \cdot 2^x$



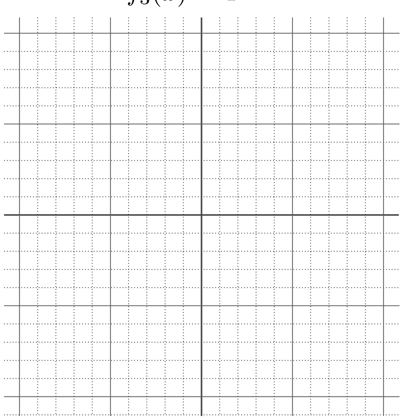
$f_2(x) = 3^{x+2}$



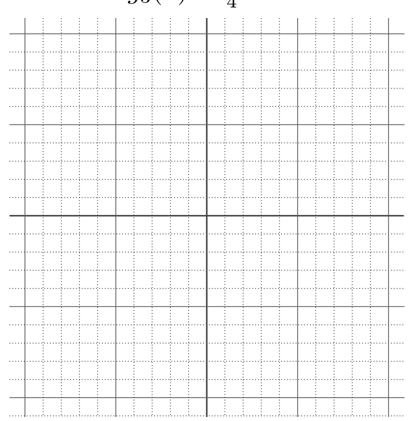
$g_2(x) = 9 \cdot 3^x$



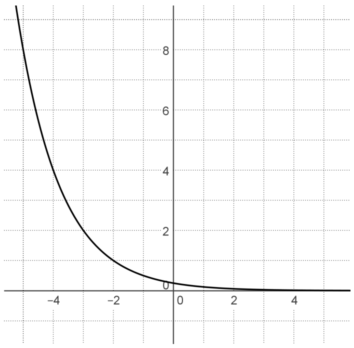
$f_3(x) = 4^{x-1}$



$g_3(x) = \frac{1}{4} \cdot 4^x$



Think About It 3.1.3 Give two different equations, similar to the pairs in Exploration 3.3, for this graph.

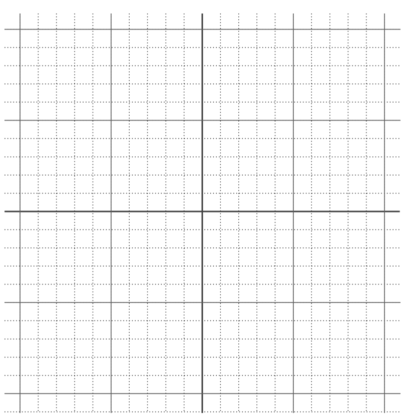


Think About It 3.1.4 Is it possible to think of the function $g(x) = 8^x$ as a transformation of the function $f(x) = 2^x$? If so, describe the transformation as precisely as you can. If not, explain why not. How would your answer change if $g(x) = 9^x$?

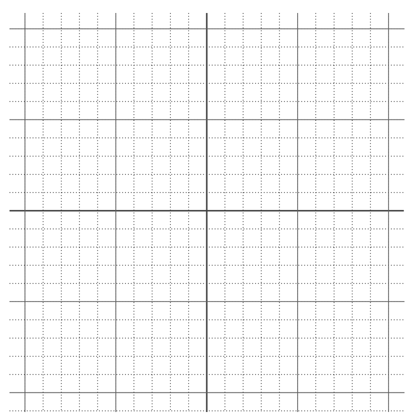
Exploration 3.4

Choose any increasing exponential function. Write an equation for your function and sketch its graph. Then sketch the graph of the function's inverse on the same set of axes and write an equation for the inverse. Repeat for a decreasing exponential function.

increasing



decreasing



Definition 3.1.2 A logarithmic function is a function of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

The two most frequently used bases for logarithms are 10 and e. The base-10 log is known as the **common log** and the base-e log is known as the **natural log**. There is a shorthand notation for each of these logs: $\log_{10} x$ can be written simply as $\log x$, while $\log_e x$ can be written as $\ln x$.

Exploration 3.5

Experiment with several different values of b in Definition 3.1.2 in order to identify some characteristics that all logarithmic functions share and to identify some characteristics that differ depending on the value of b. Be sure to think about the various properties of functions explored in Chapter 2.

Exploration 3.6

This exploration concerns expressions of the form $y = \left(1 + \frac{n}{x}\right)^x$.

E1. Without a calculator, make a prediction about the value of $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

E2. Use a calculator to check your prediction.

E3. Use a calculator to estimate the value of each limit to the nearest thousandth.

(a) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

(b) $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

(c) $\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x$

(d) $\lim_{x\to\infty} \left(1 + \frac{2.5}{x}\right)^x$

E4. Based on the results in E3 what do you expect the value of $\lim_{x\to\infty} \left(1 + \frac{n}{x}\right)^x$ to be?

Problem Set 3.1

1. For what values of b, if any, will $f(x) = b^x$ be an increasing function? If there are none, explain why.
2. For what values of b, if any, will $f(x) = b^x$ be a decreasing function? If there are none, explain why.
3. For what values of b, if any, will $f(x) = \log_b x$ be an increasing function? If there are none, explain why.
4. For what values of b, if any, will $f(x) = \log_b x$ be a decreasing function? If there are none, explain why.
5. Express the equation $\ln(9x) = \frac{3}{5}$ in exponential form and then find the value of x to the nearest thousandth.
6. Express the equation $\ln\left(\frac{x}{5}\right) = -\frac{3}{4}$ in exponential form and then find the value of x to the nearest thousandth.

In problems 7 to 12, write each function in a form that shows explicitly how it is a child of the parent function $P(x) = 2^x$.

7. $f(x) = 8^x$

8. $f(x) = 16^x$

9. $f(x) = \left(\sqrt{2}\right)^x$

10. $f(x) = 3^x$

11. $f(x) = \pi^x$

12. $f(x) = e^x$

In problems 13 to 29,

a) State the domain and range of g.
b) Give the equation of the asymptote of g.
c) Identify the intercepts of g.
d) Make a neat graph of g, plotting four points with care.
e) Use limit notation to describe the end behavior of g.
f) Find the equation of g^{-1}.

13. $g(x) = 3^x$

14. $g(x) = \left(\frac{1}{3}\right)^x$

15. $g(x) = 3^{-x}$

16. $g(x) = -3^x$

17. $g(x) = 2 \cdot 3^x$

18. $g(x) = 3^{x+1}$

19. $g(x) = \log_2 x$

20. $g(x) = \log_6 x$

21. $g(x) = \log_2(-x)$

22. $g(x) = -\log_2 x$

23. $g(x) = \log_{0.5} x$

24. $g(x) = 3 + 2^x$

25. $g(x) = 8 - 3^x$

26. $g(x) = 3^{x-5} + 2$

27. $g(x) = \log_4(x + 3)$

28. $g(x) = \log_5(x - 1)$

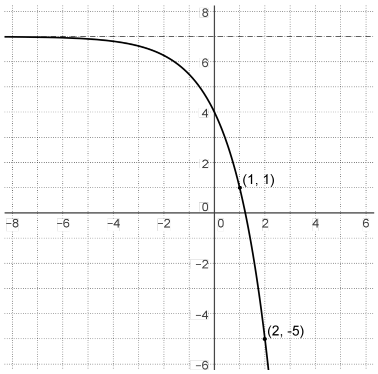
29. $g(x) = 2 + \log_4(x - 5)$

---

30. Write down an exponential or logarithmic equation which produces a graph for which the following are true: (i) the y-intercept is 2; (ii) $(-1, 5)$ and $(2, -0.25)$ are points on the graph; (iii) the graph's only asymptote is $y = -1$.

In problems 31 to 38, write an equation for the function shown.

31.



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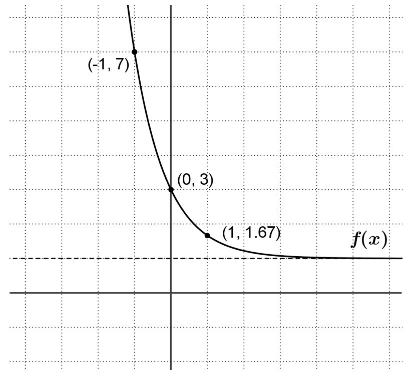

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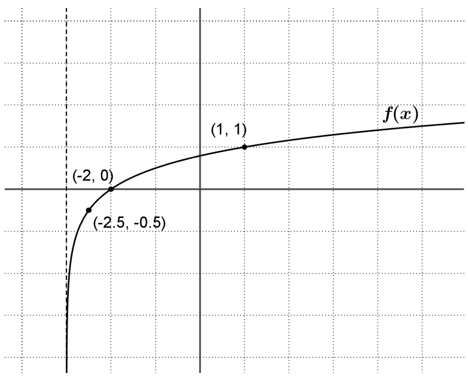

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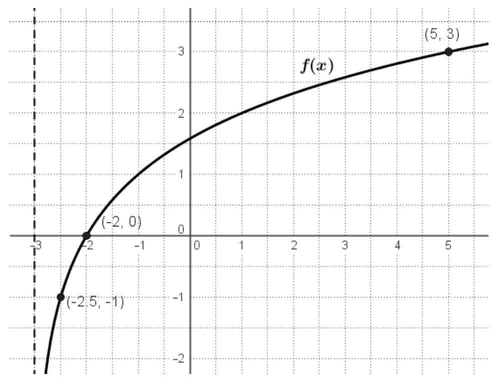

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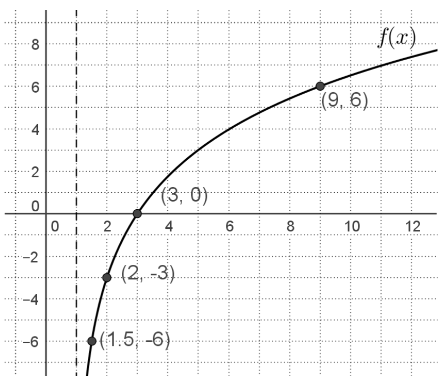

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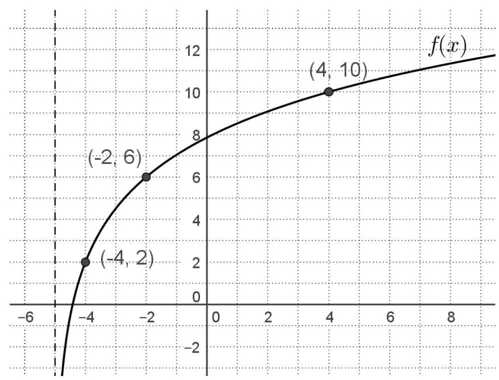

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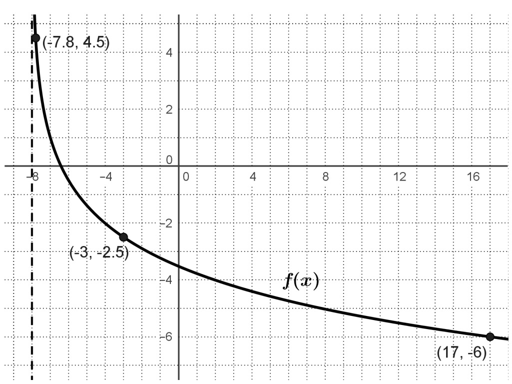
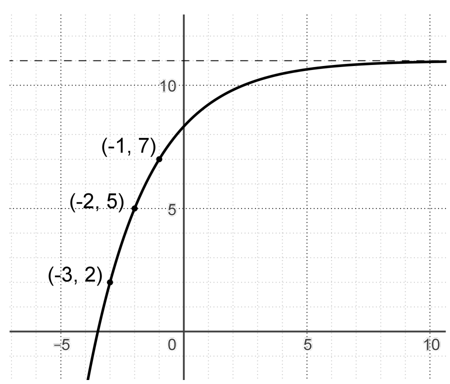


38.



39. Find a function of the form $f(x) = Ca^x$ such that the points $(1, 6)$ and $(3, 24)$ are on the graph of $f(x)$.

40. Write down both the exact value and an approximation correct to the nearest thousandth of

$$\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x.$$

41. Consider a piece of paper that is 0.004 inches thick and is folded in half, and then in half again, and so on.

a) Write an equation for a function which gives the thickness of the resulting wad of paper as a function of the number of times it has been folded.
b) What is the domain of the function you wrote in part (a)? Explain your reasoning.
c) How many folds are needed to make the wad approximately one inch thick?
d) How many folds would be needed to make the wad approximately one foot thick?
e) How many folds would be needed to make the wad approximately one mile thick?

Section 3.2 Applications of Exponential Functions

Exploration 3.7

After searching for work over the first two months of summer vacation, you finally receive two offers to start work on August 1. The first potential employer offers to pay you $100 on the first day and then give you a raise of $500 per day for each subsequent day that you work. The second potential employer offers to pay you 5¢ on the first day and to double your salary each subsequent day that you work. If you were to work every day in August, what would your salary be on August 8 and on August 31 at each job? What would the graph of salary as a function of days since August 1 look like in each case?

Quantities that change by having the same amount added to (or subtracted from) them for a particular change in time are represented by linear functions. For these quantities, how much there is at any given point in time does not affect how much they will change by in the next time period.

In contrast, when a quantity changes in such a way that for a given change in time the quantity is always multiplied (or divided) by the same value, we have an exponential function. For these quantities, the more you have, the bigger the change in a given time period will be.

•Annual Percentage Change• In news reports, it is common to hear about the percent change of some quantity in comparison to a previous time. Often, these reports give the percent change since the previous year, as in these examples from the November 2013 *New York Times* article "Homeless Tally Taken in January Found 13% Rise in New York."[1]

(A) "In New York, where the shelter population has reached levels not seen since the Depression era, the count in January estimated 64,060 homeless people in shelters and on the street in January 2013, or 13 percent more than in January 2012."

(B) "Among large cities, only Los Angeles had a larger percentage increase. Its homeless population rose by 27 percent, although its total of 53,798 was lower than New York's."

(C) "Nationwide, the number of homeless people dropped by 4 percent from 2012, to 610,042 from 633,782."

In each of these examples, the percent given is a rise over the previous year, so it is an **annual percentage change**. In the New York and Los Angeles examples, we are not told how many homeless people there were in January 2012, but we can figure it out using the given information.

Example 3.2.1 Use the information in item (A) above to determine how many homeless people there were in New York City in January 2012.

Solution First, it is crucial to note that *the 13% in this example is 13% of the unknown January 2012 number, not 13% of 64,060*! Make sure you understand why this is the case and why it matters.

If N_o represents the number of homeless people in New York in January 2012, then there will be $0.13N_o$ *more* homeless people in January 2013. And we know that in January 2013 there were 64,060 homeless people, so we can write

$$N_o + 0.13N_o = 64,060 \tag{3.1}$$

To solve for N_o, we first factor it out of the expression on the left:

$$N_o(1 + 0.13) = 64,060 \tag{3.2}$$

Finally, we can divide both sides by 1.13 and round to the nearest person to obtain the result

$$N_o = 56,690 \tag{3.3}$$

It will be helpful in understanding various forms of exponential equations going forward to pay some attention now to the expression in parentheses in eq. (3.2). **The 1 serves the purpose of *keeping* the original number of homeless people and the $+\,0.13$ serves the purpose of adding 13% of this amount.**

[1]Navarro, Mireya. "Homeless Tally Taken in January Found 13% Rise in New York." The New York Times. November 21, 2013. Accessed August 21, 2016. http://www.nytimes.com/2013/11/22/nyregion/january-tally-of-homeless-population-found-13-jump-in-city-us-data-say.html.

Think About It 3.2.1 Use the information in item (B) above to determine how many homeless people there were in Los Angeles in January 2012.

Think About It 3.2.2 Write an equation similar to eq. (3.2) which includes all of the information given in item (C) above. Is the equation true?

Exploration 3.8

Imagine the unlikely scenario where every year from January 2010 to January 2020 the annual percentage increase in New York City's homeless population was 13%. Determine New York's homeless population

E1. in January 2010

E2. in January 2020

E3. in July 2015

E4. t years after January 2010

Though annual percentage changes in various quantities are frequently reported in news stories, percentage increases over other time periods are certainly used as well. It can be useful, for the purposes of comparing percent increases given over different time periods, to determine the annual percentage change of each.

Think About It 3.2.3 A Reuters article in the April 17, 2015 *New York Times* included the line "The so-called core C.P.I., which strips out food and energy costs, increased 0.2 percent in March after a similar rise in February." If this pattern were to continue for an entire year, what would the annual percentage change in the core C.P.I. be?

•Banking• In banking, the annual percentage change in your balance if you do not put any money into or take any money out of the account is known as the **APY**, which is short for **annual percentage yield**. If the bank compounds interest annually, this is the same value as what banks call the **APR**, which is short for **annual percentage rate**, but if the compounding is more often, the APY will be higher than the APR. If a bank compounds semiannually, instead of calculating your interest once a year at the APR, they award interest twice a year, determining how much to give you each time by multiplying your balance by half the APR. If a bank compounds quarterly, they award interest four times a year, determining how much to give you each time by multiplying your balance by one quarter of the APR.

Think About It 3.2.4 Explain why the APY is higher than the APR if interest is compounded more than once a year.

Example 3.2.2 While today you'd be hard pressed to find a U.S. bank offering even a 2% APR, back in the early 1980s, APRs above 10% were common. (Of course, banks also charged much higher interest rates for those borrowing money in the 80s than they do today.) Find

(a) the APY if the APR is 12% and interest is compounded semiannually

(b) the amount you would have in the bank after 5 years of this interest scheme if you had deposited $1000 initially

Solution

(a) While it's not absolutely necessary to introduce variables to answer this first question, we'll be glad to have them when we get to the next part, so let's call the initial balance A_o and let $A(t)$ be the amount in the account after t years. After half a year, we keep the initial amount, and add on 6% of it, so we have

$$A(0.5) = A_o(1 + 0.06)$$

and after another half year, we keep the amount we had after the first interest payment,

$A(0.5)$, and add on 6% of this amount, so we have

$$\begin{aligned} A(1) &= A(0.5)(1+0.06) \\ &= A_o(1.06)(1.06) \\ &= A_o(1.06)^2 \\ &= 1.1236A_o \end{aligned}$$

Since the amount after 1 year is 1.1236 times the original amount, we know that our annual percentage change or APY is 12.36%. This means that a 12% APR compounded semiannually is equivalent to an annual compounding rate of 12.36%.

(b) If we think of this as getting 6% of our account balance twice a year for five years, we might write

$$\begin{aligned} A(5) &= 1000\left(1.06^2\right)^5 \\ &= 1000(1.06)^{10} \\ &= 1790.85 \end{aligned}$$

Alternatively, we can think of this of an annual percentage change of 12.36% for five years, in which case we might write

$$\begin{aligned} A(5) &= 1000(1.1236)^5 \\ &= 1790.85 \end{aligned}$$

Either way, we arrive at the conclusion that there will be a balance of $1790.85 after five years.

Think About It 3.2.5 Write an expression which can be evaluated to find $A(t)$ after t years if an amount A_o is deposited in an account that earns 12% interest compounded

(a) monthly (b) daily (c) n times per year

Think About It 3.2.6 What do you think happens to the amount of money in your account as the value of n in item (c) of TAI 3.2.5 approaches infinity? That is, if you imagine compounding every minute or every second or every millisecond or every picosecond or …

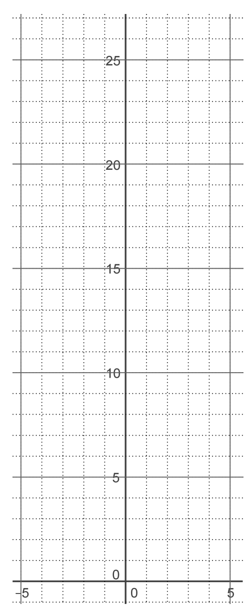
Think About It 3.2.7 What is the APY for an investment with an APR of 12% compounded continuously, as described in TAI 3.2.6? What is the APY for the general case in which the APR is r and interest is compounded continuously?

Think About It 3.2.8 Explain the relationship between the graphs of $y = a(1.1274969)^x$ and $y = ae^{0.12x}$.

•Growth by a Constant Factor• Often in a situation involving exponential growth we may think in terms of the time it takes for a quantity to change by a constant factor rather than the percent change in the quantity in a given time. It is vital to understand that this is just a different way of conceptualizing exponential growth, *not* a different type of growth. Biologists, for example, are frequently interested in the doubling time of population.

Exploration 3.9

Lactococcus lactis is a bacterium that is important in the production of cheese and other dairy products. In a certain culture of this bacteria, there are three thousand bacteria in the culture at noon and the number of bacteria present doubles every hour. Let $f(t)$ represent the number of bacteria in thousands as a function of the number of hours after noon.



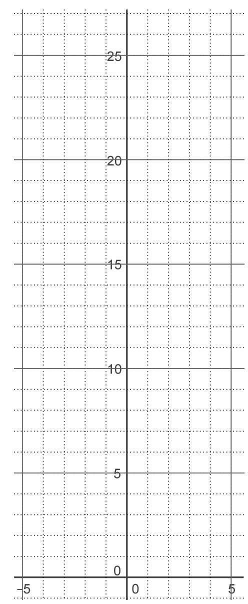
E1. Find $f(0)$, $f(1)$, $f(2)$, $f(3)$ and $f(4)$. Then graph $y = f(t)$.

E2. What is $f(-1)$ and what information does it give you about the situation?

E3. Determine the equation for $f(t)$. Explain why your equation makes sense.

Imagine you are a researcher in the lab with this bacteria culture that has three thousand bacteria at noon, but that you actually began looking at the culture at 10:00 am, so you choose to describe the bacterial population using a function $g(t)$ which gives the number of bacteria in thousands **as a function of the number of hours after 10:00 am**.

E4. Graph $y = g(t)$ and find its equation.



E5. Compare your graph of $y = g(t)$ to the graph of $y = f(t)$ above. What transformation of f results in g?

E6. There is a second transformation of f that will also result in g. What is it? Can you explain why these two transformations produce the same result?

Think About It 3.2.9 What is the hourly percentage growth rate of a population that doubles every hour? How about for a population that triples every hour?

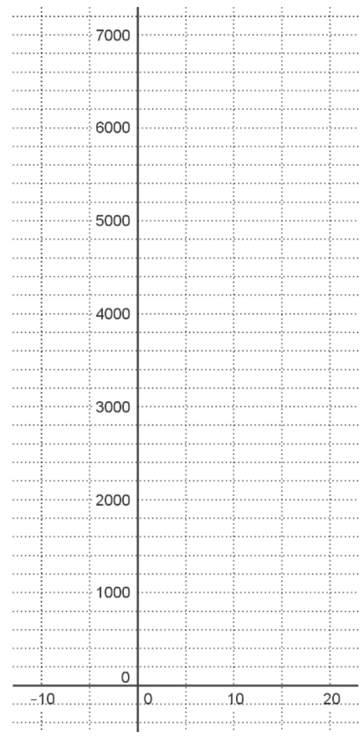
The opening paragraph from a 2015 article from OutdoorHub offers another example of discussing exponential growth in terms of a constant factor rather than a annual percent increase:

> For the first time in decades, mountain lions are reclaiming territory in southern and eastern Alberta. Half a century ago, the wild cats were pushed into the province's remote corners, but conservation efforts and good habitat conditions are now drawing cougars back to their former range. According to the CBC, Alberta's mountain lion population has grown from 680 animals a decade ago to more than 2,000 now, a nearly threefold increase.[2]

Exploration 3.10

To model the situation discussed in the article just quoted, assume that the increase in the mountain lion population has been exactly 3-fold in 10 years and that the growth will continue at this (exponential) pace.

E1. Make a graph of the mountain lion population in Alberta as a function of number of years since 2005 (a decade prior to the publication date of the article) and write an equation for your graph.



E2. What transformations to the graph of $f(x) = 3^x$ would produce your graph?

E3. The article goes on to mention that a biologist had recently told a local paper that "there could be as many as $3,000$ cougars in Alberta before the end of the decade." Use your model to evaluate this claim.

E4. Write your model in a form that explicitly shows the annual percentage increase in the population.

•The Continuous Growth Model• When we want to emphasize that a quantity is changing continuously rather than at discrete time intervals, we may choose to write our exponential equation describing the growth in the form

$$A(t) = A_o e^{rt} \tag{3.4}$$

[2] Xu, Daniel. "Alberta's Mountain Lion Population Triples, Human Encounters on the Rise." OutdoorHub. April 13, 2015. Accessed August 20, 2016. http://www.outdoorhub.com/news/2015/04/13/albertas-mountain-lion-population-triples-human-encounters-rise/.

and we refer the value of r in this case as the **continuous growth rate** or the **relative growth rate**. (In a banking context, we can, of course, continue to refer to it as an APR or annual rate and specify that the compounding is continuous.)

As we saw in TAI 3.2.8, we don't *have to* write our exponential equation in the form $A(t) = A_o e^{rt}$ just because we know the change is continuous. There is still an annual percentage change in the quantity A, and we can write an equation that produces the same graph in the form

$$A(t) = A_o(1 + r)^t \tag{3.5}$$

It is essential to remain aware, however, that while A, t, and A_o represent the same thing in both forms of the equation, ***r* represents different things in eq. (3.4) and eq. (3.5)**. Assuming that time is measured in years, the r in eq. (3.4) is the continuous (or relative) annual growth rate of A, while the r in eq. (3.5) is the annual percentage change in A.

(In situations where the exponential model is not likely to be useful on a time scale of years, say in a laboratory experiment, a different unit of time may be chosen as the standard. If, for example, times used in the equation are in hours, the r in eq. (3.4) would be the continuous hourly growth rate, while the r in eq. (3.5) would be the hourly percentage change.)

Think About It 3.2.10 Determine the continuous growth rate that will produce an annual percentage change of 25%. (Before doing any calculation, determine whether your answer should be higher or lower than 0.25.)

One situation which can be well modeled by an exponential equation and where change is continuous is that of radioactive decay. While scientists could have chosen to build up tables of values of annual (or hourly or …) percentage changes or continuous growth rates for the myriad radioactive elements, they have instead chosen to make tables of their **half-lives**. The half-life of a radioactive element is the time it takes for the mass of a sample of the element to be reduced by half.

Example 3.2.3 The element lawrencium-265 has a half life of 10 hours. Write an equation for $A(t)$, the mass of a sample of lawrencium-265 as a function of the time t (in hours) since it had a mass of A_o.

Solution One approach is to begin with eq. (3.4). While we don't know the continuous rate of decay, we do know that $A(10) = 0.5A_o$. (Why?)

$$\begin{aligned} A(10) &= A_o e^{r\cdot 10} \\ 0.5A_o &= A_o e^{10r} \\ 0.5 &= e^{10r} \\ \ln 0.5 &= 10r \\ r &= \frac{\ln 0.5}{10} \\ r &\approx -0.0693147 \end{aligned}$$

So one form of the equation for $A(t)$ is

$$A(t) = A_o e^{-0.0693147t} \tag{3.6}$$

Alternatively, we could begin with eq. (3.5), and again use the fact that $A(10) = 0.5A_o$:

$$\begin{aligned} A(10) &= A_o(1+r)^{10} \\ 0.5A_o &= A_o(1+r)^{10} \\ 0.5 &= (1+r)^{10} \\ 0.5^{1/10} &= 1+r \\ r &\approx -0.066967 \end{aligned}$$

So another form of the equation for $A(t)$ is

$$A(t) = A_o(1-0.06697)^t \tag{3.7}$$

But there is yet another approach to writing an equation when you're given the half-life, one which doesn't require any calculation at all. It requires recognizing that if time equal to two half-lives has passed, there will be $\frac{1}{4}$ of the original amount remaining, while if time equal to three half-lives has passed, there will be $\frac{1}{8}$ of the original amount remaining, and if time equal to n half-lives has passed, there will be $\left(\frac{1}{2}\right)^n$ of the original amount remaining. In other words, we can think of this as change by a constant factor, as in the bacteria and mountain lion explorations above. For the case we've been considering in this example, this yields the equation

$$A(t) = A_o\left(\frac{1}{2}\right)^{t/10} \tag{3.8}$$

Think About It 3.2.11 Explain why the t in the exponent of eq. (3.8) is divided by the half-life.

Think About It 3.2.12 Verify that eq. (3.6), eq. (3.7), and eq. (3.8) all give the same value for A when $t = 25$.

Think About It 3.2.13 Write an equation a form similar to that of eq. (3.8) in order to model the population of a bacteria colony which has 1000 bacteria initially and triples in size every 14 hours.

Think About It 3.2.14 Explain what each of the letters in the equation $y = a \cdot C^{t/d}$ tells you about the situation being modeled.

Each form of an exponential function that we've considered in this section is just a variation on the basic form $f(t) = a \cdot b^t$. In each case, we have a base raised to our input variable and then multiplied by a constant. *All that varies from one form to another is how we choose to write the base.* We choose the way to write the base by thinking about what's most convenient or what information we'd like to emphasize in the particular situation we're modeling.

Think About It 3.2.15 For each of the following common forms of an exponential equation, write down the expression which will be equivalent to b when the equation is written in the form $A(t) = a \cdot b^t$ and indicate what the various parameters in your expression for b tell you about the context in which the equation is being used.

$A(t) = A_o(1 + r)^t$

$A(t) = A_o\left(1 + \frac{r}{n}\right)^{nt}$

$A(t) = A_o e^{rt}$

$A(t) = A_o \cdot C^{t/d}$

Problem Set 3.2

1. A February 2016 *New York Times* article by Mireya Navarro begins, "Evictions in New York City dropped last year to their lowest level in a decade, as the administration of Mayor Bill de Blasio bolstered efforts to prevent more New Yorkers from becoming homeless. Evictions decreased by 18 percent last year, to $21,988$ from $26,857$ in 2014."[3]

a) Quantitatively inclined readers Xavier and Yesenia attempt to check Navarro's math. Xavier writes $21,988(1+0.18) \stackrel{?}{=} 26,857$, and Yesenia writes $26,857(1-0.18) \stackrel{?}{=} 21,988$. Assuming that the actual numbers of evictions are correct,[4] what do you think they conclude? What's going on?

b) The article goes on to say "Evictions dropped in 2014, though only by 6.9 percent." How many evictions does this imply that there were in 2013?

c) Navarro goes on to tell us that the number of evictions in 2013 was $28,849$. Assuming this value is accurate, evaluate Navarro's claim that evictions dropped by 6.9 percent from 2013 to 2014.

2. A May 2016 *New York Times* article by Adam Nagourney[5] begins,

> The homeless population in Los Angeles County jumped 5.7 percent last year, with a sharp increase in tents and homeless encampments offering daily evidence of the problem sweeping this region, county officials said Wednesday. Yet the findings, based on a three-night block-by-block census of homeless people living on the street, also described reason for optimism: a 30 percent drop in the number of homeless veterans and an 18 percent decrease in homeless families.

The article goes on to say,

> The overall homeless population increased from 44,359 in January 2015 to 46,874 in the count in January 2016 in all of Los Angeles County. . . . The number of homeless veterans in the county dropped from 4,362 in the 2015 count to 3,071 this year; the number of homeless families slipped from 8,103 to 6,611 this year."

Use the data provided by Nagourney to evaluate his claims about the percent changes in the overall homeless population, the number of homeless veterans, and the number of homeless families. If you object to his claims, what percent values would you use?

3. The Dow Jones Industrial Average declined by 5.295% in the month of January 2014. If the value at the end of January was 15,698.85, determine

a) the value (to the nearest hundredth) at the start of January 2014

b) the value (to the nearest hundredth) at the end of December 2014 in the (unlikely) scenario that the average were to continue to decline by 5.295% every month

4. Over the course of the day on Tuesday, January 13, 2015, Exxon Mobil stock lost 2.13% of its value.

a) If the price at the stock at the end of the trading day was $90.89, what was its price (to the nearest cent) at the start of the trading day?

b) If the stock had continued to decline by the same daily percentage for the rest of the trading week, what would its price (to the nearest cent) have been at the end of the day on Friday, January 16?

[3] Navarro, Mireya. "Evictions Are Down by 18%; New York City Cites Increased Legal Services." The New York Times. February 29, 2016. Accessed August 21, 2016. http://www.nytimes.com/2016/03/01/nyregion/evictions-are-down-by-18-new-york-city-cites-increased-legal-services.html.

[4] Yearly data from city marshal reports is available at http://cwtfhc.org/evictions-marshals-documents/. Navarro provides a link to a summary of this data in her article.

[5] Nagourney, Adam. "Homelessness Rises in Los Angeles, Except for Veterans and Families." The New York Times. May 04, 2016. Accessed August 21, 2016.http://www.nytimes.com/2016/05/05/us/los-angeles-homelessness-veterans-families.html.

5. A news report noted that Twitter's stock value at the close of business on Tuesday, February 25, 2014 was \$54.96, which was down by 1.47% from its value at the close of business on Monday, February 24.

a) Find the closing value (to the nearest cent) of Twitter's stock value on Monday, February 24.
b) In the (unlikely) scenario that the stock were to continue to decline by 1.47% each day of the week, determine the closing value of Twitter's stock (to the nearest cent) on Friday, February 28.

6. Explain the difference between APR and APY.

7. If \$50,000 is invested at an APR of 6%, find the value of the investment after 7 years to the nearest cent if the interest is compounded

a) annually b) quarterly c) monthly d) continuously

8. If \$8,000 is invested at an APR of 5.2%, find the value of the investment after 9 years to the nearest cent if the interest is compounded

a) annually b) semi-annually c) daily d) continuously

9. Determine the APY (to the nearest ten-thousandth of a percent) on an investment with an interest rate of 5.8% compounded

a) quarterly b) monthly c) daily d) continuously

10. Consider the equation $g(x) = 100\left(1 + \frac{.08}{4}\right)^{4x}$. Write an equation in each of the following forms that would have a graph identical to the graph of g. Use Desmos or another graphing utility to verify that you function is the same as g.

a) $y(x) = ab^x$
b) $y(x) = a(1+r)^x$
c) $y(x) = ae^{rx}$
d) $y(x) = a \cdot 2^{x/c}$
e) $y(x) = a \cdot 3^{x/c}$
f) $y(x) = a\left(1 + \frac{r}{12}\right)^{12x}$

11. If the following equation is true, arrange a, b, c, and d in order from smallest to largest without actually finding their values:

$$17e^{at} = 17\left(1 + \frac{b}{2}\right)^{2t} = 17\left(1 + \frac{c}{4}\right)^{4t} = 17\left(1 + d\right)^{t}$$

12. Consider the equation $f(t) = 2^t$, where t is measured in months.

a) If the equation is rewritten in the form $f(t) = (1+r)^t$, what is the value of r when expressed as a percent? What does r represent?
b) If the equation is rewritten in the form $f(t) = e^{rt}$, make an estimate of what the approximate value of r will be when expressed as a decimal. Explain your choice. The find the exact value of r. What does r represent in this equation?
c) Find the daily and yearly growth rates of f.
d) If $g(t) = e^{at}$ models the same situation as f, but with t measured in days, find the value of a.

13. At the beginning of an experiment there are 10 fruit flies. In 15 days there are 640. Write an equation for the population of fruit flies as a function of the number of days since the beginning of the experiment, assuming the population growth is exponential. Use your equation to determine how many fruit flies there will be after 21 days.

14. An experimenter observes that a population of fruit flies in an experiment doubles every 2.5 days. Write an equation for the population of fruit flies as a function of the number of days since the beginning of the experiment if there were 10 fruit flies initially. Use your equation to determine how many fruit flies there will be after 21 days.

15. An experimenter observes that a population of fruit flies in an experiment triples every 3.962 days. Write an equation for the population of fruit flies as a function of the number of days since the beginning of the experiment if there were 10 fruit flies initially. Use your equation to determine how many fruit flies there will be after 21 days.

16. An experimenter checks a population of fruit flies each morning and determines that the population increases by 31.95% each day. Write an equation for the population of fruit flies as a function of the number of days since the beginning of the experiment if there were 10 fruit flies initially. Use your equation to determine how many fruit flies there will be after 21 days.

17. An experimenter expects the relative growth rate of a population of fruit flies to be 27.73% if time is measured in days. If he begins with a population of 10 fruit flies, write an equation which predicts the population of fruit flies as a function of the number of days since the beginning of the experiment. Use your equation to determine how many fruit flies the experimenter expects to see after 21 days.

18. If the population of a town decreased from 18,000 in 1970 to 10,000 in 1990, write an exponential equation which would provide a model of the population as a function of the number of years since 1970.

19. In 1900, the population of the United States was roughly 76 million and by 1950 the population had nearly doubled.

a) Create an exponential model with a doubling time of 50 years as a parameter that will give the US population in millions as a function of the number of years since 1900. What population, to the nearest person, does your model predict for the year 2010?

b) Use your model (and your calculator and your knowledge of, say, logarithms) to predict the year in which the population will reach $500,000,000$.

20. According to the US census, the US population in 1900 was actually 76,212,168. In 1950, the population was actually 151,325,798. As you'll note from these above numbers, the US did not exactly double in 50 years as you had been asked to assume in problem 19.

a) What is the ratio of the population in 1950 to the population in 1900, rounded to the nearest thousandth?

b) Create an exponential model with your answer from part (a) and the given population from 1900 as parameters that will give the US population as a function of the number of years since 1900. What population, to the nearest person, does your model predict for the year 2010?

c) Note that your graphing calculator or Desmos is perfectly capable of using the unrounded ratio of the given populations instead of the rounded answer you got for part (a). Use THAT number in your model to predict the population in 2010.

d) Between this problem and problem 19, you've now used three different models to predict the population in 2010. What are the advantages/disadvantages of the three models? Is one model better than the others? Why do you think so?

21. What if the populations in 1900 and 1950 from problem 20 had been reversed?

a) Create an exponential model for this fictional situation and use it to predict the US population in 2010.

b) Use your model to predict the year in which the population will reach 12.79 million (the 2017 population of Pennsylvania).

22. Robert finds 200 grams of Machemersium, a radioactive substance with a half-life of 42 years.

a) Create a model for the amount of Machemersium Robert will have t years from now.

b) How much Machemersium does your model predict that Robert (or his descendants) will have 200 years from now?

c) How much Machemersium does your model tell you there was 1000 years before Robert found it?

23. The population of a Machemersia (a small country, its main export is Machemersium; see problem 22) has grown about 1.5% every year.

a) If Machemersia had a population of 1,000,000 people at the start of the year 1992, how many people would it have one year later? Two years later? Three years later?

b) Use your numbers from part (a) to come up with a function that models the population of Machemersia as a function of time in years since 1992.

c) Use your model to predict Machemersia's population at the start of the year 2017.

24. The population of a Robertistan (a small country, its main export is Precalcula; see problem 25) has declined in population by about 2% every year.

a) If Robertistan had a population of 1,000,000 people at the start of the year 1980, come up with a function that models the population of Machemersia as a function of time in years since 1980.

b) Use your model to predict Robertistan's population at the start of the year 2017.

25. Precalcula, a not-so-newly-discovered precious element, is known to decay radioactively, but no one has yet bothered to determine its half-life. In 1974, 500.0 grams of it were found. By 2000, there were 100.0 grams of the sample remaining.

a) Write an equation that gives the mass of the sample as a function of the number of years since 1974.

b) How much of the original sample, to the nearest tenth of a a gram, would be left in 2017?

c) Use your model to predict how long would it take a sample of 9000 grams to decay to 100 grams.

d) What is the half-life of Precalcula?

26. A certain Toyota Prius, purchased for \$26,000 in 2007, was worth only \$7000 by 2017.

a) Write an exponential equal that models the value of the car as a function of the number of years since 2007.

b) What does your model predict the car will be worth in 2037?

c) How reasonable is your model's prediction for part (b)? (Are there mitigating factors that might affect how well your model works over time?)

27. Rewrite the equation $A(x) = 300e^{-0.042x}$ in the form $A(x) = A_o\left(\frac{1}{2}\right)^{x/c}$. Then describe a situation that these equations might model. For each equation, indicate specifically what all of the variables and numbers would represent in the situation you've described.

28. In 1900 North Carolina's population was 1.89 million, and by 2010 its population was 9.54 million. Assuming that the state's population as a function of time can be reasonably modeled by an exponential function, find the doubling time of the population (to the nearest tenth of a year) and the annual percentage change in the population (to the nearest hundredth of a percent).

29. The half-life of terbium-148 is one hour and there are 21.00 g of a sample of terbium-148 at noon.

a) Write an equation that gives the mass of the sample as a function of t, the number of hours after noon.
b) How much of the sample, to the nearest hundredth of a gram, was present at 9:30 a.m.?
c) How much of the sample, to the nearest hundredth of a gram, will be present at 3:15 p.m.?
d) At what time, to the nearest minute, will there be 0.50 g remaining?

30. The half-life of beryllium-11 is 13.81 seconds. How long (to the nearest hundredth of a second) will it take for a 30.00 mg sample to decay to 1.00 mg?

31. The half-life of fluorine-18 is 109.77 minutes. How long (to the nearest hundredth of a minute) will it take for a 500.00 mg sample to decay to 5.00 mg?

32. Consider the function $p(t) = 1000e^{0.1t}$

a) Carefully describe a situation that this equation could model. Be as precise as you possibly can about what the variables and constants represent in your imagined context.
b) What is the growth rate per unit of time that you've chosen for t? Rewrite the equation in a form that shows this growth rate explicitly.
c) What is the doubling time? Rewrite the equation in a form that shows the doubling time explicitly.
d) Verify that all of your equations give the same result to a reasonable number of decimal places for $p(50)$.

33. An equation modeling the growth of a certain bacteria culture is $P(t) = 12000 \cdot 3^{t/14}$, where P is the bacteria population t hours after noon on February 10.

a) What does the 14 in this equation tell you about the situation?
b) What was the population to the nearest whole number at 8 a.m. on February 10?
c) Rewrite the equation in the form $P(t) = P_0 e^{rt}$
d) Determine the daily growth rate of the population to the nearest hundredth of a percent.

34. An equation modeling the growth of a certain bacteria culture is $P(t) = 8000 \cdot 10^{t/90}$, where P is the bacteria population t hours after noon on March 12.

a) What does the 90 in this equation tell you about the situation?
b) What was the population (to the nearest whole number) at 7 a.m. on March 12?
c) Rewrite the equation in the form $P(t) = P_0 e^{rt}$
d) Determine the daily growth rate of the population to the nearest hundredth of a percent.

35. An equation modeling the growth of a certain bacteria culture is $P(t) = 1600 \cdot 5^{t/9}$, where P is the bacteria population t hours after 11:00 a.m. on November 12.

a) What does the 9 in this equation tell you about the situation?
b) What was the population at 4 a.m. on November 12?
c) What time, to the nearest minute, will it be when the population reaches 2100?
d) Determine the hourly growth rate of the population to the nearest hundredth of a percent.
e) Determine the daily growth rate of the population to the nearest hundredth of a percent.
f) Find P_o and a if the equation modeling the bacteria's growth is written in the form $P(t) = P_o \cdot 2^{t/a}$.

Section 3.3 Quadratic Functions

Definition 3.3.1 A **quadratic function** is a function which can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

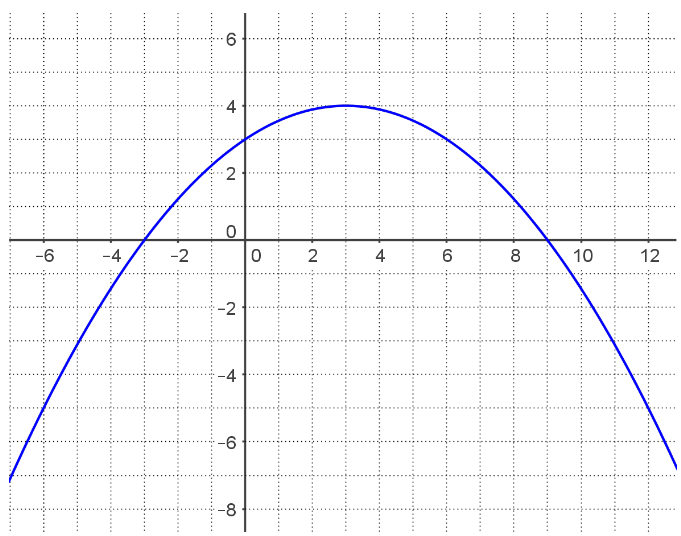
Think About It 3.3.1 Why is the restriction on the value of a necessary in Definition 3.3.1?

Think About It 3.3.2 Is a quadratic function a power function?

Exploration 3.11

Consider the following forms in which one might opt to write the equation for a quadratic function and the graph of the quadratic function shown:

- $f(x) = ax^2 + bx + c$
- $f(x) = a(x-h)^2 + k$
- $f(x) = a(x-z_1)(x-z_2)$
- $f(x) = a(x-m)(x-n) + p$
- $f(x) = ax(x-q) + c$



E1. Determine how to write the equation for the function whose graph is shown in as many of the given forms as you can.

E2. Write some other quadratic equation in one of the given forms and try to figure out how to express it in each of the other forms.

E3. Can any quadratic equation be written in any of the forms? If so, how do you know? If not, give some examples of quadratic functions which can't be written in one or more of the given forms.

E4. Try to write a description of an algebraic process for transforming the first form into one of the other forms. How many of the forms can you do this for?

E5. For each form, find the y-intercept and the x-coordinate of the vertex in terms of the given parameters.

Problem Set 3.3

1. Consider the function $f(x) = -4x^2 + 12x - 10$.

a) Write the equation of f in the form $f(x) = Ax(x + B) + C$
b) Make a rough sketch of the graph of f, labeling the vertex, the y-intercept, and one other point with their coordinates.
c) Write the equation of f in x-intercept form.

2. Find n if the graph of $g(x) = 3x^2 + nx + 2$ has a minimum value of 0.

In problems 3 to 8, write $Q(x)$ in x-intercept form and in vertex form.

3. $Q(x) = 2x^2 + 5x + 3$

4. $Q(x) = 3x^2 + 4x + 1$

5. $Q(x) = 5x^2 + 7x + 2$

6. $Q(x) = x^2 + 2x + 3$

7. $Q(x) = x^2 - x + 4$

8. $Q(x) = -2x^2 + 12x + 20$

In problems 9 to 20,

a) Identify the intercepts of f.
b) State the domain and range of f.
c) Make a neat graph of f on graph paper, plotting five points, including the vertex, with care.

9. $f(x) = 2(x + 3)^2 - 4$

10. $f(x) = -2(x - 1)^2 + 3$

11. $f(x) = -(x - 3)(x + 5)$

12. $f(x) = 2(x - 5)(x + 7)$

13. $f(x) = -2(x + 4)(x - 6)$

14. $f(x) = x^2 - 4x + 13$

15. $f(x) = 2x^2 - 8x + 8$

16. $f(x) = 10 + 6x - x^2$

17. $f(x) = 13 + 4x - x^2$

18. $f(x) = x^2 + 10x + 14$

19. $f(x) = 2x^2 + x + 3$

20. $f(x) = 3x^2 + 12x + 5$

In problems 21 to 24, write an equation for the parabola with the indicated features.

21. y-intercept of 6; x-intercepts of -2 and 12

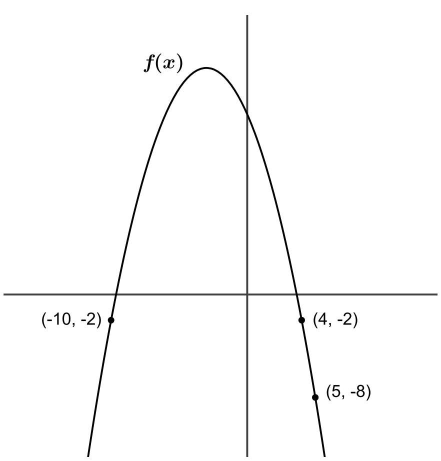
22. y-intercept of -3; x-intercepts of 1 and -5

23. y-intercept of -12; x-intercepts of 3 and 6

24. y-intercept of -12; vertex of $(3, 15)$

In problems 25 to 28, write an equation for the quadratic function shown.

25.



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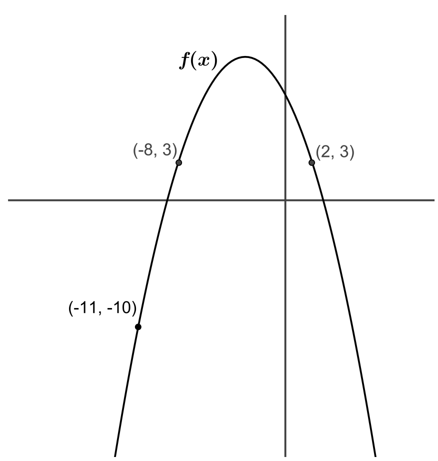

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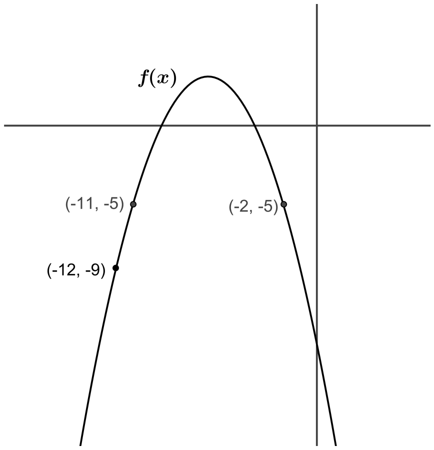

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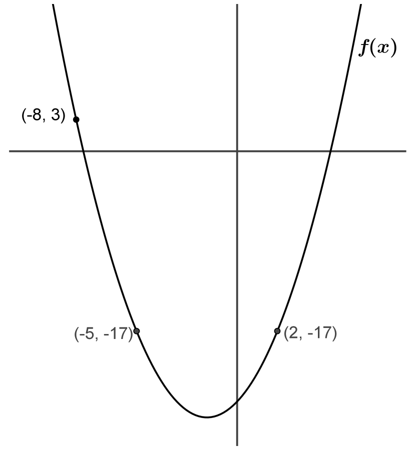


29. A ball is tossed up into the air out of a window. The ball's height above the ground as function of time is given by the equation $h(t) = 12 + 32t - 16t^2$. When does the ball reach its highest point and how high is it at that time?

30. A ball is tossed up into the air out of a window. The ball's height above the ground as function of time is given by the equation $h(t) = 50 + 40t - 16t^2$. When does the ball reach its highest point and how high is it at that time?

31. A slingshot launches a pebble up into the air out of a window. The pebble's height above the ground as function of time is given by the equation $h(t) = 11 + 48t - 16t^2$. When does the pebble reach its highest point and how high is it at that time?

32. The GS varsity Scrabble team holds a rally at 5:00 pm. At 5:06, 61 Cougar Crazies are there, and at 5:42, 133 Cougar Crazies are there. The rally ends at 6:30 pm.

a) Assuming the number of Cougar Crazies at the rally can be modeled by a quadratic function and that the peak attendance occurs at 5:42, write an equation for $Q(t)$, the number of Cougar Crazies at the rally as a function of the number of minutes that have passed since 5:00 pm.
b) What is the y-intercept of the graph of $y = Q(t)$ and what does it tell you about the situation?
c) State the domain and range of Q.

33. Find a value for p such that the function $g(x) = \frac{1}{2}x^2 + 8x + p$ has a minimum value of 1.

34. Find a value for p such that the function $g(x) = \frac{1}{3}x^2 + 4x + p$ has a minimum value of 1.

35. Find a value for p such that the function $g(x) = -\frac{1}{4}x^2 + 3x + p$ has a maximum value of 10.

36. Explain, using your knowledge of the quadratic formula, or by any other method that occurs to you, why a quadratic function which has a root of $p + qi$ must also have a root of $p - qi$. Then find an equation in vertex form of the quadratic function whose leading coefficient is 1 and whose roots are $p + qi$ and $p - qi$. How are the roots related to function's graph?

Section 3.4 Polynomial Functions

Definition 3.4.1 A **polynomial function** is a function which can be written in the form

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

where n is a whole number. The **degree** (or **order**) of the polynomial is the largest value of i for which $a_i \neq 0$. This value of a_i is known as the **leading coefficient**.

Think About It 3.4.1 While it must be *possible* for a function to be written in the form indicated in Definition 3.4.1 in order to be classified as a polynomial, we don't typically encounter polynomials written using sigma notation. Write out the expanded form of the polynomial function which has a degree of 4 and has $a_0 = -5$, $a_1 = 2$, $a_2 = 0$, $a_3 = 1$, and $a_4 = \frac{1}{2}$.

Think About It 3.4.2 Give some examples of functions that are not polynomial functions.

Think About It 3.4.3 Try to write a definition for a polynomial function that doesn't use sigma notation.

Think About It 3.4.4 Is $f(x) = 0$ a polynomial function? If so, what does the definition tell you about its degree? If not, explain why it doesn't qualify as a polynomial according to the definition.

Think About It 3.4.5 Use Desmos (or some other graphing utility) to look at the end behavior of a variety of polynomial functions. What patterns do you notice?

Exploration 3.12

Consider polynomials of the form $P(x) = a(x-b)^m(x-c)^n$. (Remember that m and n must be whole numbers in order for P to be a polynomial.) Using Desmos or similar graphing technology, explore what happens to the graph of the function near its zeros as you change the values of the various parameters of P. Summarize your results below.

Exploration 3.13

Consider the question of how to factor the polynomial $P(x) = x^4 + 2x^3 - 5x^2 - 4x + 6$.

E1. If you think only about the leading coefficient and the constant term of P, determine which of the following binomials have the potential to be factors of P: $(x - 1)$, $(x + 2)$, $(x - 4)$, $(2x + 1)$

E2. Again considering only the leading coefficient and the constant term of P, and limiting yourself to relatively prime integer values of a and b, list all other binomials of the form $(ax - b)$ that could be factors of P. What does this tell you about the possible rational roots of P?

E3. If $(ax-b)$ is a factor of $P(x)$, what must be true about $P\left(\frac{b}{a}\right)$? Use this fact and the results of the previous parts to identify one binomial factor of $P(x)$ and to eliminate some of the binomials you previously identified as possible factors.

E4. Use the long division algorithm to divide $P(x)$ by the binomial factor you identified in the previous part and then use a process similar to the one you've just gone through to find a binomial factor of the resulting quotient.

E5. Write down the factored form of $P(x)$.

E6. Find the factored form of the polynomial $6x^4 + x^3 + 4x^2 + x - 2$.

Polynomials are the subject of the **Fundamental Theorem of Algebra**, a key theorem in the history of mathematics. This theorem is stated in different ways in different sources. For example, one formulation is that every polynomial of degree greater than zero can be written as a product of linear and quadratic factors. (It's worth noting that the constants and coefficients in the factors need not be integers.) Another way you'll find it stated is that a polynomial of degree n, where $n > 0$, has exactly n complex roots, so long as you count double roots as two roots, triple roots as three roots, and so on. (Here its worth noting that the real numbers are a subset of the complex numbers, so this definition allows for all of a polynomial's roots are real.)

Exploration 3.14

Consider the following polynomials through the lens of the two different statements of the Fundamental Theorem of Algebra in the previous paragraph. What observations can you make? How do the two given formulations of the Fundamental Theorem of Algebra get at the same idea in different ways?

$f(x) = x^4 - x^3 - 6x^2$

$g(x) = x^4 - 16$

$h(x) = x^3 + 8$

Problem Set 3.4

1. If f is a polynomial, state its degree and leading coefficient. If it is not a polynomial, explain why.

a) $f(x) = \sqrt{2}$

b) $f(x) = x(5-2x^2)^3$

c) $f(x) = \displaystyle\sum_{k=0}^{2} \frac{x^{5-k}}{k+3}$

d) $f(x) = 3x + \dfrac{1}{x^{-2}}$

e) $f(x) = 4\sqrt{x^3}$

f) $f(x) = \displaystyle\sum_{k=-1}^{-4} k^2 \cdot x^{6-k}$

g) $f(x) = \frac{1}{4}x(5-3x)^2(4-2x)^3$

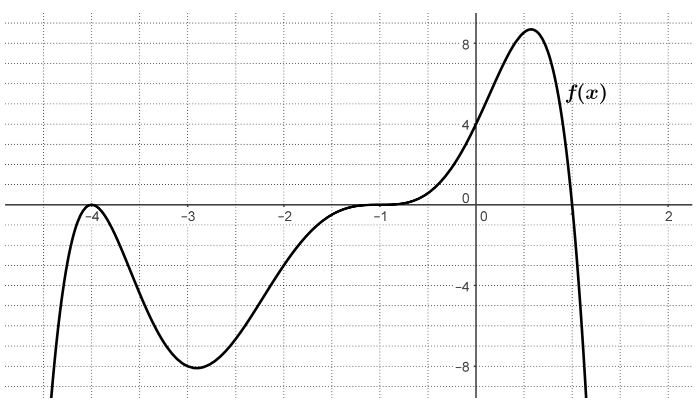
h) $f(x) = \dfrac{1}{\pi} + \pi$

2. Consider $f(x) = (x+2)(x-3)(x+5)$.

a) Sketch a quick graph of f. (Get the x- and y-intercepts, end behavior, and basic shape right.)
b) Use your graph to determine the interval(s) on which $f(x)$ is positive.

In problems 3 to 8 write an equation for the polynomial function.

3.



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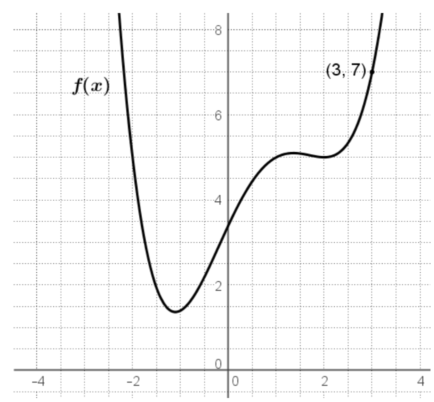

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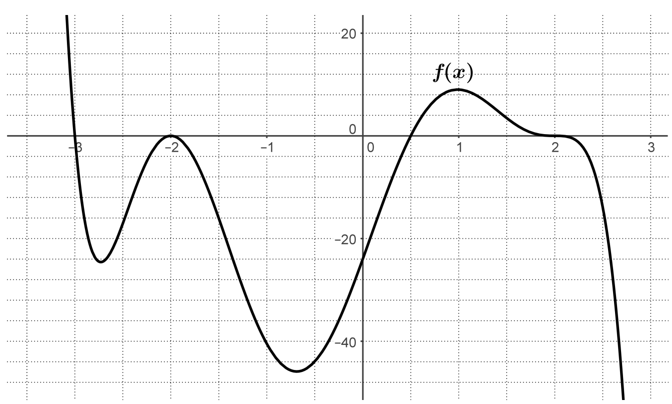

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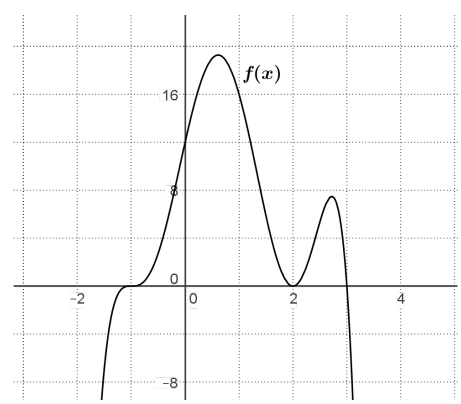

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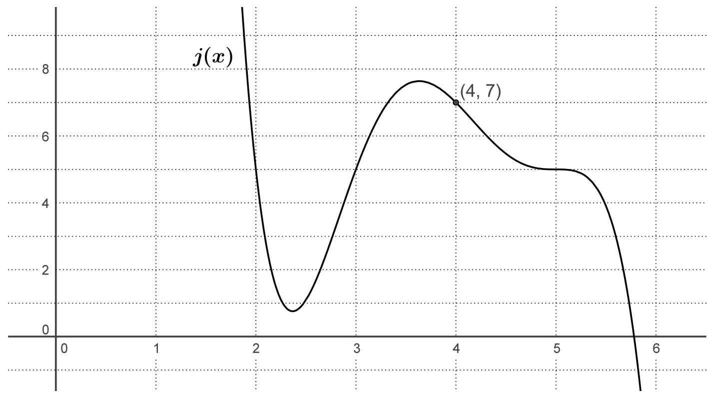

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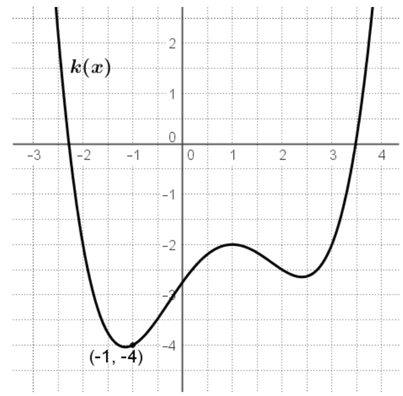


9. Using the graph for problem 6, find all values of x for which $f(x)$ is less than or equal to 0.

10. If f is the function graphed in problem 3, find the domain of g if

a) $g(x) = \sqrt{f(x)}$ b) $g(x) = 2\sqrt{-f(x)}$ c) $g(x) = \sqrt{10f(x)}$

11. If j is the function graphed in problem 7, find the domain of $g(x) = \dfrac{x+2}{j(x)-5}$

In problems 12 to 15,

a) make a rough sketch of the graph of the polynomial on the left side of the inequality and then show the solution to the inequality on a number line.

b) use a sign chart to solve the inequality algebraically and state the solution in interval form.

12. $(x+5)(x+3)^2(x-1)(x-2)^3 > 0$

13. $x(2x+1)^4(x-2)^5(x-6) < 0$

14. $-x(x+3)^2(2x+5)^3(x-1)^4 \leq 0$

15. $-5(1-x)(x-2)(7x-3)^3(x+4)^7 \geq 0$

In problems 16 to 19, without first graphing the function, write down the limit expressions which describe its end behavior.

16. $f(x) = (2-x^3)(3-x)$

17. $f(x) = x(3-x^3)(2+x)$

18. $g(x) = x^3 + 4x^5 - x^6$

19. $g(x) = 16x^2 + 4x^3 - 2x^5$

---

20. Find all zeros of $y = x^3 + 4x^2 - 8$. Start by using your calculator to plug in possible rational zeros until you find one. Then use division and an understanding of quadratic equations to find the others.

21. Graph $y = x^3 + 2x^2 + 8x + 16$ with the help of technology. According to the Fundamental Theorem of Algebra, how many zeros does this cubic polynomial function have? How many do you see on your graph (and what is the shape of the graph around those zeros – does it go through them, bounce, etc.)? What does this tell you about the remaining zeros? Find the remaining zeros.

22. Consider the function $f(x) = x^3 - 2x^2 - 4x + 8$.

a) Find one integer zero of f.
b) If a is the zero you found, divide f by $x - a$ and use the result to help you factor f completely.
c) Sketch a rough graph of $y = f(x)$. The intercepts and end behavior should be accurate, but the rest can be approximate.

23. Consider the function $f(x) = x^3 - 3x - 2$.

a) Find one integer zero of f.
b) If a is the zero you found, divide f by $x - a$ and use the result to help you factor f completely.
c) Sketch a rough graph of $y = f(x)$. The intercepts and end behavior should be accurate, but the rest can be approximate.

24. A box (with no lid) is to be formed by cutting an identical square out of each corner of a piece of 11-inch by 17-inch paper. What is the maximum volume (to the nearest hundredth of a cubic inch) that such a box could have?

25. A box (with no lid) is to be formed by cutting an identical square out of each corner of a sheet of paper which is 15 inches wide and 18 inches tall. Write an equation which expresses the volume of such a box as a function of the length of the side of a cut-out square. Then determine the maximum volume (to the nearest hundredth of a cubic inch) that such a box could have.

26. In Europe, the common paper size is called A4 and is 210 mm wide by 297 mm tall. A box (with no lid) is to be formed by cutting an identical square out of each corner of a sheet of A4 paper. Write an equation which expresses the volume of such a box as a function of the length of the side of a cut-out square. Then determine the maximum volume (to the nearest hundred cubic mm) that such a box could have.

27. Enter the function $f(x) = x^4 + 17x^3 + 78x^2 + 140x + 140$ into a graphing utility. To be sure that you've entered the function correctly, verify that, to the nearest hundredth, $f(-3.9) = 3.30$.

a) Make a nice graph on graph paper of the function. Include a few (just a few!) well-spaced and informatively labeled tick marks, be sure that the end behavior is appropriate and that key "wiggles" are clearly shown. Label intercepts and local extrema with coordinates, correct to two places after the decimal.
b) State the domain and range of f.

28. Enter the function $f(x) = -x^4 + 32x^3 - 305x^2 + 994x - 490$ into a graphing utility. To be sure that you've entered the function correctly, verify that, to the nearest thousandth, $f(0.6) = 3.382$..

a) Make a nice graph on graph paper of the function. Include a few (just a few!) well-spaced and informatively labeled tick marks, be sure that the end behavior is appropriate and that key "wiggles" are clearly shown. Label intercepts and local extrema with coordinates, correct to two places after the decimal.
b) State the domain and range of f.

29. Enter the function $f(x) = 297 - 158x + 31x^2 - x^3$ into a graphing utility. To be sure that you've entered the function correctly, verify that, to the nearest hundredth, $f(2.2) = 88.79$.

a) Make a nice graph on graph paper of the function. Include a few (just a few!) well-spaced and informatively labeled tick marks, be sure that the end behavior is appropriate and that key "wiggles" are clearly shown. Label intercepts and local extrema with coordinates, correct to two places after the decimal.
b) State the domain and range of f.

Section 3.5 Rational Functions

Definition 3.5.1 A **rational function** is a function which can be written in the form $f(x) = \dfrac{P(x)}{Q(x)}$, where P is any polynomial function and Q is any non-zero polynomial function.

Think About It 3.5.1 Are all, some, or no polynomial functions also rational functions? Explain.

Think About It 3.5.2 Consider the function $f(x) = \dfrac{x}{x+1}$. What is $f(0)$? $f(1)$? $f(2)$? $f(3)$? $f(10)$? $f(100)$? $f(1000)$? What does this suggest about $\lim\limits_{x\to\infty} f(x)$?

Think About It 3.5.3 How does changing the constant and/or the coefficients in the function from TAI 3.5.3 change the limit? Why? If someone were to give you a random number, L would it always be possible to find a combination of constant and coefficients such that $\lim\limits_{x\to\infty} f(x) = L$?

Exploration 3.15

Let $P(x) = 1$ and choose any non-constant linear function as $Q(x)$. On graph paper, make neat graphs of $y = P(x)$ and $y = Q(x)$. Then use the quotient of the y-coordinates at a number of x-values on these graphs to construct the graph of $f(x) = \dfrac{P(x)}{Q(x)}$. Be sure to use a number of non-integer x-values near the zero of Q. (Why?) Note the domain and range of f.

Repeat using a quadratic function with real zeros for Q.

Repeat using non-constant linear functions for both P and Q.

Repeat using a non-constant linear function for P and a quadratic function with real zeros for Q.

Repeat using a quadratic function with real zeros for P and a non-constant linear function for Q.

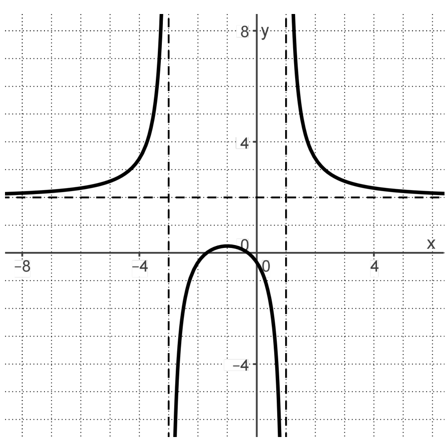
Repeat using a quadratic function with no real zeros for P and a non-constant linear function for Q.

Repeat using $P(x) = x^2 - 3x - 4$ and $Q(x) = x + 1$

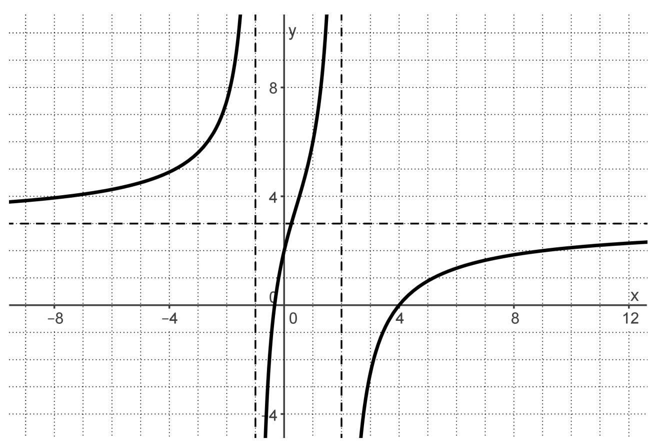
As you look over your graphs, what seems worth commenting on? What generalizations can you make? What questions might be worth exploring further?

Problem Set 3.5

1. If $f(x)$ is the function graphed below, write the limit expressions needed to fully describe the behavior of f near its asymptotes.



2. If $f(x)$ is the function graphed below, write the limit expressions needed to fully describe the behavior of f near its asymptotes.



In problems 3 to 6, show the solution to the given inequality on a number line.

3. $\dfrac{(x+5)(x+3)}{x-1} > 0$

4. $\dfrac{2x(x-1)}{(x+2)(2x+3)} < 0$

5. $\dfrac{-2(x+1)}{(x-4)^2} \leq 0$

6. $\dfrac{3(x-1)^2(x-5)}{(4-x)(x+2)^3} \geq 0$

In problems 7 to 12, write an equation for a rational function which has the features described.

7. a vertical asymptote of $x = 1$, an x-intercept of 7, and a hole at $x = -2$

8. a horizontal asymptote of $y = 0$ and a hole at $x = 2$

9. a horizontal asymptote of $y = 3$ and a y-intercept of -5

10. a vertical asymptote of $x = -1$, an x-intercept of 3, and a y-intercept of 5

11. a domain of $(-\infty, \infty)$ and a horizontal asymptote of $y = 0$

12. a domain of $(-\infty, \infty)$ and a horizontal asymptote of $y = -\dfrac{1}{2}$

In problems 13 to 18, find the intercepts, the x- and y-coordinates of any holes, and the equations of all horizontal and vertical asymptotes of the function. If any of these do not exist, explain how you know.

13. $f(x) = \dfrac{2x+3}{x^2+6x+8}$

14. $f(x) = \dfrac{x^2+6x+8}{2x+3}$

15. $g(x) = \dfrac{x^2-4}{9x^2+1}$

16. $g(x) = \dfrac{5x-20}{2x(x-4)}$

17. $h(x) = \dfrac{1-4x^2}{x^2-5}$

18. $h(x) = \dfrac{x^3}{5x(x-4)}$

In problems 19 to 22, graph the given function without the help of technology. Take care to accurately plot the intercepts, holes, and asymptotes. State the domain and, if possible, the range. If it is impossible to determine the range without the help of technology explain why. Use graphing technology to check your result and then rethink anything that doesn't look right.

19. $f(x) = \dfrac{x^2-6x+5}{x-5}$

20. $f(x) = \dfrac{x^2-7x+6}{2x-2}$

21. $f(x) = \dfrac{x^2-3x-4}{x^2-16}$

22. $f(x) = \dfrac{2}{x+2}$

---

23. Explain why the function $f(x) = \dfrac{6x^2+13x+2}{2x+5}$ must have a slant asymptote. Then find the equation of this asymptote.

Section 3.6 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

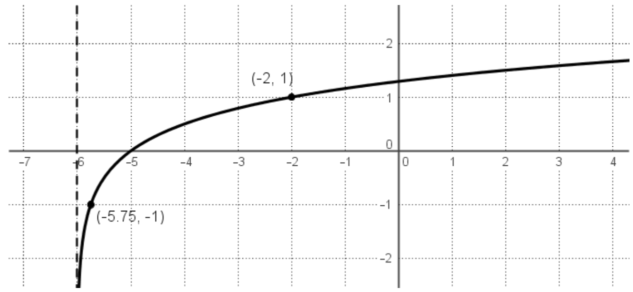
Problem Set 3.6

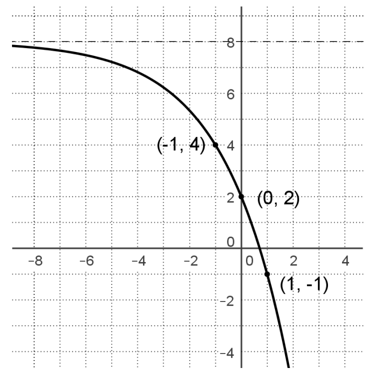
1. Find the exact value of a if the function $f(x) = 12^x$ is written in the form $f(x) = 3^{ax}$, and then write the function in this form using an approximation of a which is correct to three places after the decimal.

2. Consider the function $g(x)$ graphed at right.

a) Write an equation for g.

b) Write limit expressions to describe the end behavior of g.





3. Consider the function $f(x)$ graphed at right.

a) Write an equation for f.

b) Write limit expressions to describe the end behavior of g.

4. Make a careful graph of $h(x) = -\log_3(x+6)$, plotting four points and any asymptotes with care. Label each asymptote with its equation and label your four points with their coordinates.

5. A bank offers an interest rate of 4.5% compounded monthly. Find

a) the annual percentage yield (APY), to the nearest thousandth of a percent
b) the balance after 7 years, to the nearest cent, for an initial investment of $1500

6. A radioactive isotope, which has a mass of 1.552 g at 8:00 PM on January 1 decays in such a way that its mass decreases by 13.7% each hour.

a) Write an equation which gives the sample's mass as a function of the number of hours after 8:00 PM.
b) What was the mass (to the nearest thousandth of a gram) of the sample at 1:00 PM on January 1?
c) What day and time will it be, to the nearest minute, when the sample has a mass of 0.200 grams?

7. Consider the equation $f(x) = 80\,(0.5)^{x/7.600}$

a) Describe as specifically as you can, a situation that this equation could be modeling. Your description should describe what the various numbers and variables stand for and should include units.
b) Rewrite the given equation in the form $f(x) = a \cdot e^{bx}$, expressing b as a number correct to three significant figures.
c) What information does your value for b in part (b) provide about the situation you described in part (a)?

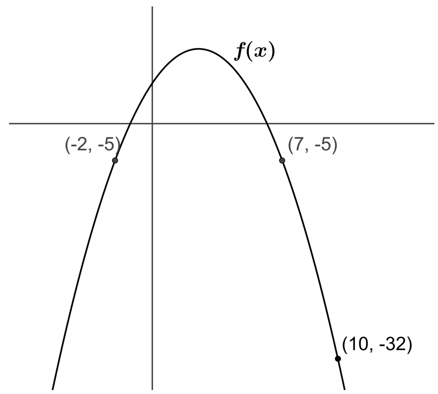
8. From 1920 to 1950 the population of Puerto Rico grew from 1.300 million to 2.211 million

a) Write an exponential equation (in any form you like) that gives the population of Puerto Rico as a function of the number of years since 1920. (Parameters should be expressed as decimals correct to four significant figures.)
b) Use your function to estimate the year in which the population was just 500,000.
c) Determine the annual percentage increase and the tripling time for your population function. Give answers to four significant figures.

9. Consider the function $g(x) = 3x^2 + 2x + 1$.

a) If g is rewritten in the form $a(x-b)^2 + c$, find the values of a, b, and c.
b) If g is rewritten in the form $p(x-q)(x-r)$, find the values of p, q, and r.
c) State the domain and range of g.

10. Write an equation for the quadratic function f graphed at right.



11. A person standing on top of a building throws a ball up into the air off the top of a building so that it eventually falls to the ground below. The ball's height (in feet) above the ground as function of time (in seconds) since it was released is given by the equation $h(t) = 50 + 64t - 16t^2$.

a) When does the ball reach its highest point?
b) How far above the ground is the ball when it's at its highest point?
c) Roughly how tall is the building?

12. Determine which of the following are polynomials. For those that aren't, explain why they fail to satisfy the definition of a polynomial For each one that *is* a polynomial, state the *degree* and the *leading coefficient*.

$a(x) = 6$

$b(x) = x^2\sqrt{5} - \dfrac{x}{2}$

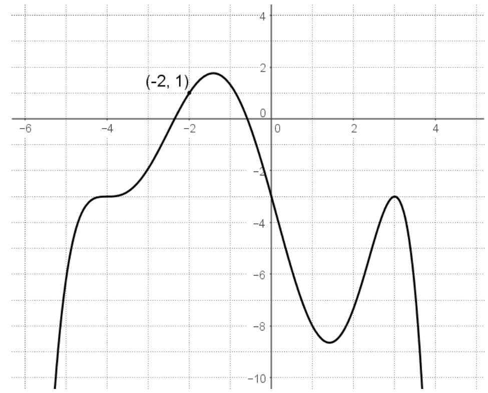
$c(x) = \displaystyle\sum_{i=0}^{3} 5i \cdot x^{2i}$

$d(x) = 3x^{-2} + 4x^{-3} + 5x^{-4}$

13. Consider the polynomial $P(x) = a(x-2)^b(3-x)(2x-7)^2$. Write down one combination of values for a and b that will make $\lim\limits_{x\to-\infty} P(x) = \infty$ and $\lim\limits_{x\to\infty} P(x) = -\infty$. Explain why the values you chose will work.

14. Consider the rational function $R(x) = \dfrac{f(x)}{8x^2 - x + 5}$. Find a function $f(x)$ so that R will have a horizontal asymptote of $y = \dfrac{3}{2}$ and an x-intercept of 2.

15. Write an equation for the polynomial function f graphed at right.

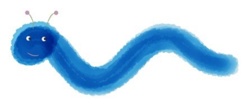


16. Show the solution to the inequality $\frac{1}{4}(x+6)^3(3x-1)^2(x+2) \leq 0$ on a number line.

17. Solve $\dfrac{(x+6)^3(3x-1)^2}{4(x+2)} \leq 0$ algebraically. Note the similarity to problem 16. Why does its difference affect your final answer but not the work necessary to solve the inequality?

4 Circular Functions

Section 4.1 Girt



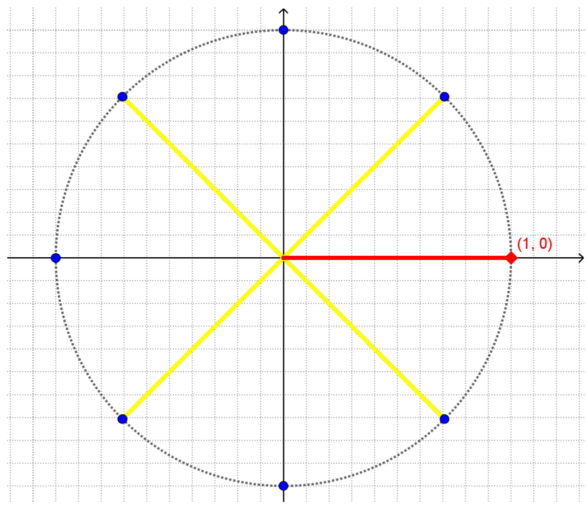
Perhaps you've heard of Girt the Worm, perhaps not. At any rate, at the moment she finds herself in the middle of a field full of tasty flowers but inexplicably stuck at the end of a long, rigid, very light rod due east of a post around which the rod pivots as Girt wiggles.

Exploration 4.1

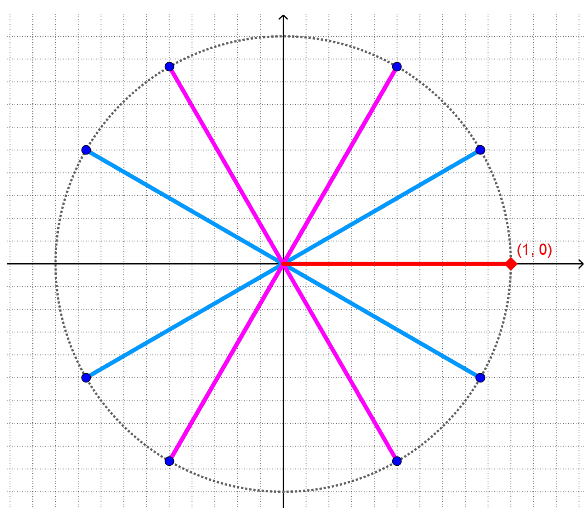
There are flowers at each of the locations A-Z listed below. Each flower's position is specified by an East/West coordinate and a North/South coordinate in relation to the post on which Girt's rod pivots. Without using a calculator, classify as many flowers as you can according to whether they are ones that Girt will encounter (and can, therefore, eat) or will not encounter as she wriggles along, stuck at the end of her long, rigid, rotating rod. Begin by identifying those which are easiest to classify.

A	1	rod	E	$\frac{1}{2}$	rod	N
B	$\frac{3}{4}$	rod	W	$\frac{4}{5}$	rod	S
C	0.3420	rod	E	0.7232	rod	N
D	0.5403	rod	E	0.8415	rod	N
E	$\frac{\sqrt{2}}{3}$	rod	E	$\frac{\sqrt{7}}{3}$	rod	S
F	$\frac{\sqrt{2}}{3}$	rod	W	$\frac{\sqrt{3}}{3}$	rod	N
G	$\frac{\sqrt{15}}{4}$	rod	W	$\frac{1}{4}$	rod	S
H	$\frac{\sqrt{0}}{2}$	rod	E	$\frac{\sqrt{4}}{2}$	rod	N
I	$\frac{\sqrt{3}}{2}$	rod	E	$\frac{\sqrt{2}}{2}$	rod	N
J	1.1220	rod	W	0.6431	rod	N
K	$\frac{5}{13}$	rod	E	$\frac{12}{13}$	rod	S
L	$\frac{4}{3}$	rod	E	$\frac{5}{3}$	rod	N
M	0.9900	rod	W	0.1411	rod	N
N	1	rod	W	0	rod	N
O	$\frac{4}{5}$	rod	W	$\frac{3}{5}$	rod	S
P	$\frac{\sqrt{1}}{2}$	rod	E	$\frac{\sqrt{3}}{2}$	rod	N
Q	0.4161	rod	W	0.9093	rod	N
R	$\frac{\sqrt{4}}{3}$	rod	E	$\frac{\sqrt{5}}{3}$	rod	S
S	0.5807	rod	E	0.6207	rod	N
T	0.2837	rod	E	0.9589	rod	S
U	$\frac{\sqrt{3}}{2}$	rod	E	$\frac{1}{2}$	rod	N
V	$\frac{\sqrt{14}}{4}$	rod	W	$\frac{\sqrt{2}}{4}$	rod	S
W	1	rod	W	1	rod	N
X	$\frac{\sqrt{2}}{2}$	rod	E	$\frac{\sqrt{2}}{2}$	rod	N
Y	$\frac{\sqrt{5}}{5}$	rod	W	$\frac{2}{\sqrt{5}}$	rod	S
Z	0.7071	rod	W	0.8660	rod	N

Think About It 4.1.1 Here's a bird's eye view (from the eye of a bird who is not tempted by worms) of Girt's world. Girt's starting point at the end of her rod is labeled with the coordinates (1, 0) to represent a position 1 rod due east of the post on which her rod pivots. Notice that in this view the circle along which Girt can travel has been divided into 8 equal sectors. Which of the points shown correspond to flowers listed in Exploration 4.1? If our unit of measure is rod lengths, how far must Girt travel from her starting point to reach each of the points shown, assuming she travels in a counter-clockwise direction?



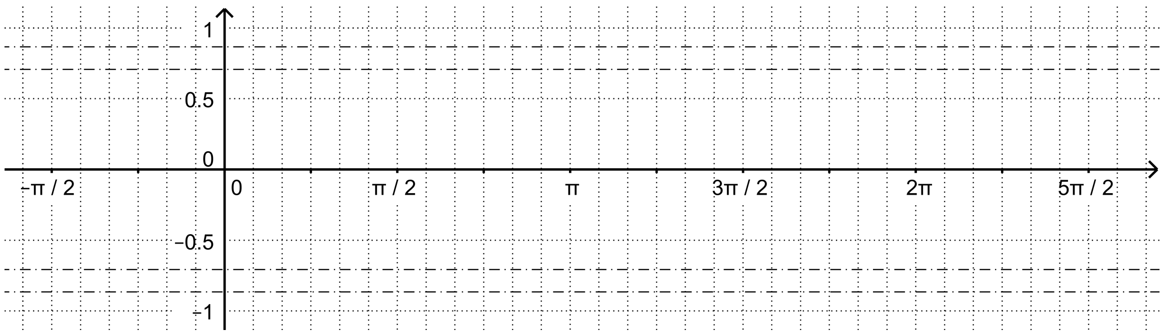
Think About It 4.1.2 Here's another birds eye view, from the eye of a bird whose vision divides the circle into 12 equal sectors. Which of the points shown correspond to flowers listed in Exploration 4.1? How far (in rod lengths) must Girt travel from her starting point to reach each of the points shown, assuming she travels in a counter-clockwise direction?



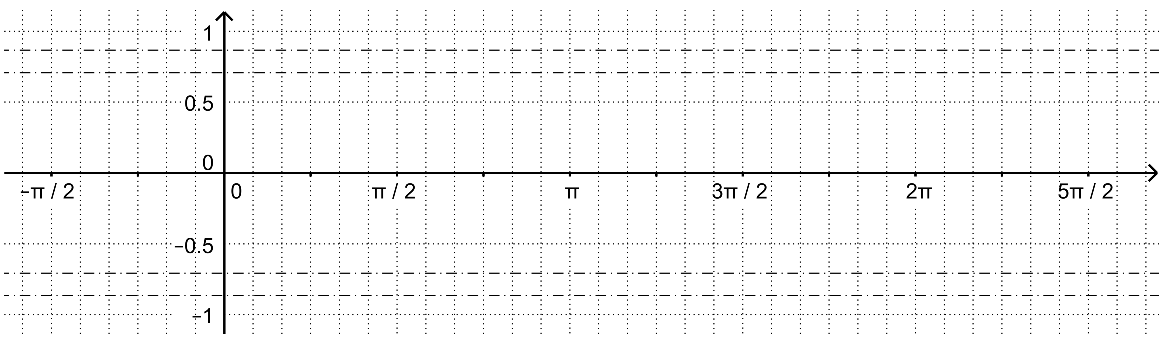
Think About It 4.1.3 Identify the coordinates of each point shown in TAI 4.1.1 and TAI 4.1.2.

Now imagine a function $x(d)$ where the input is the distance Girt has traveled from her starting point along the circle in a counter-clockwise direction, and the output is her resulting x-coordinate in a bird's eye view. Before she has gone anywhere, the distance she has traveled is 0 and her x-coordinate is 1, so the point $(0,1)$ is on the graph of $x(d)$. When she has gone a quarter of the way around the circle, the distance she has traveled is $\frac{\pi}{2}$ rod lengths and her x-coordinate is 0, so the point $\left(\frac{\pi}{2},0\right)$ is on the graph of $x(d)$. When she has gone a twelfth of the way around the circle, the distance she has traveled is $\frac{\pi}{6}$ rod lengths and her x-coordinate is $\frac{\sqrt{3}}{2}$, so the point $\left(\frac{\pi}{6},\frac{\sqrt{3}}{2}\right)$ is on the graph of $x(d)$.

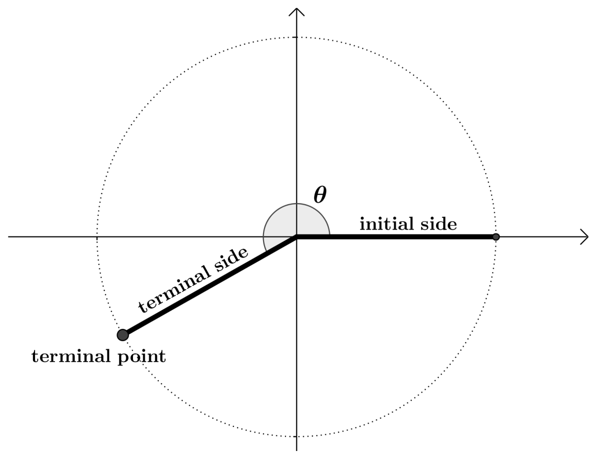
Think About It 4.1.4 By plotting the points just described, and similar points obtained using the distances and coordinates you found in TAIs 4.1.1 to 4.1.3, make a graph of the function $x(d)$, Girt's x-coordinate as a function of counterclockwise distance traveled.



Think About It 4.1.5 Now graph $y(d)$, Girt's y-coordinate as a function of counterclockwise distance traveled.



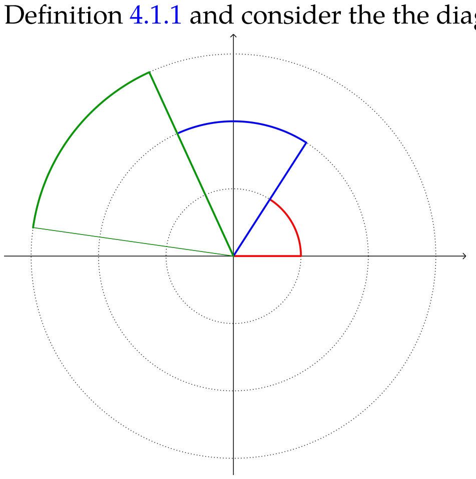
Think About It 4.1.6 Describe how the vertical grid lines on the graphs in TAI 4.1.4 and TAI 4.1.5 are related to the radial lines in the bird's eye views in TAI 4.1.1 and TAI 4.1.2.



In TAI 4.1.4 and TAI 4.1.5 we considered the functions' inputs to be the *signed distance* Girt had traveled along the circle, but we could have considered the inputs to be the *signed angle* which subtended the arc along which she traveled. In measuring distance, we always started at the point $(1,0)$ and considered motion in a counterclockwise direction to be positive. In measuring angles, we'll assume that the **initial side** of the angle Girt is creating by wriggling along to rotate the rod is the positive x-axis. When Girt pauses somewhere, her rod creates the angle's **terminal side** and Girt herself is located at the **terminal point** associated with that angle. Measuring in a counterclockwise direction from the initial side to the terminal side gives us a positive measure for the angle created, while measuring in a clockwise direction from initial side to the terminal side gives us a negative measure. We can also get angle measures greater than 360°(or less than $-360°$) by imagining going around the circle more than once before stopping at the terminal side. It is worth noting, however, that it is quite rare to measure angles in degrees in higher math. Rather, we use a measure which, unlikely the arbitrary degree, is connected in a fundamental way to the nature of the circle: the radian.

Definition 4.1.1 A **radian** is defined as the measure of a circle's central angle which subtends an arc whose length is equal to the circle's radius.

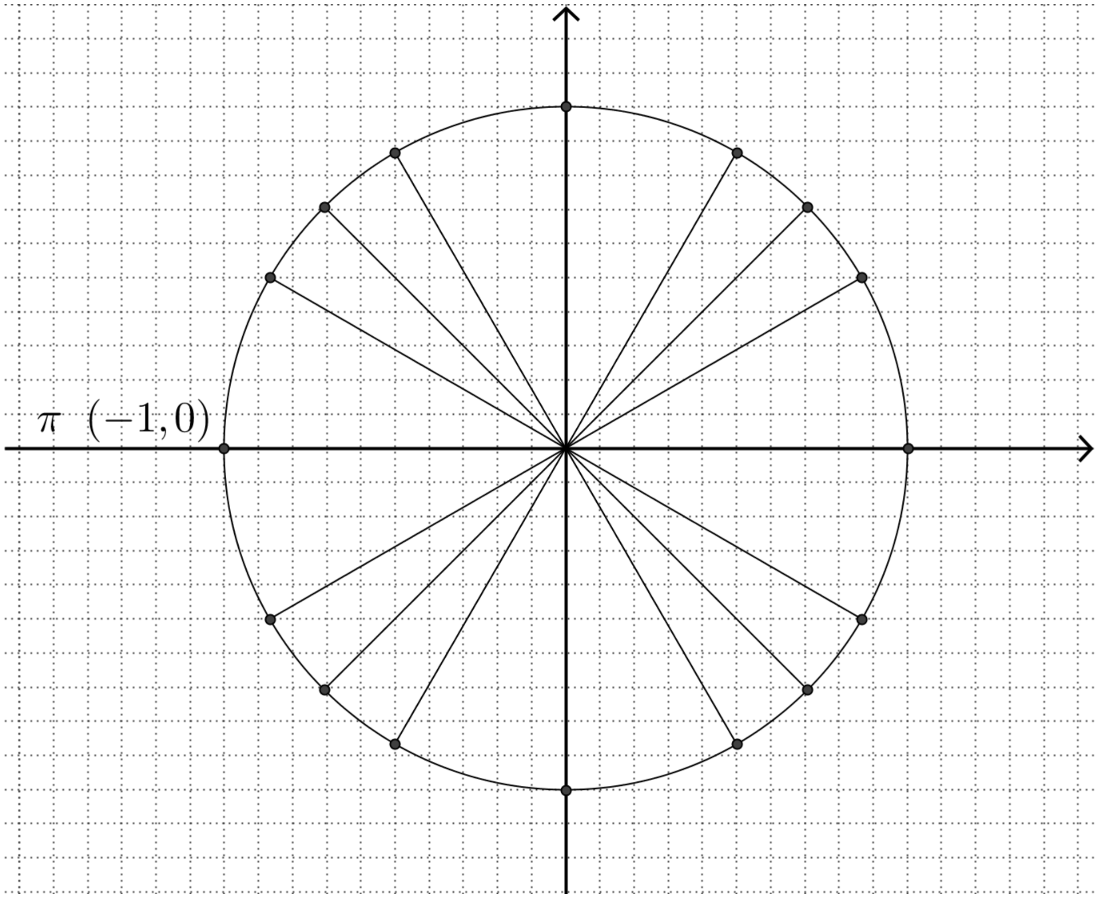
Think About It 4.1.7 Write down an insight or a question or two that you have as you read Definition 4.1.1 and consider the the diagram below in connection with it.



Think About It 4.1.8 How does the signed distance in rods (*i.e.*, radii) associated with Girt's travel to any point on a unit circle compare to the signed angle (in radians) associated with her trip to the point? Explain.

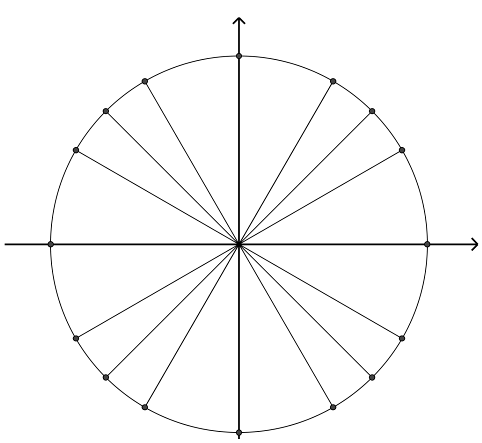
Problem Set 4.1

1. Next to each point marked with a dot, write the point's exact coordinates and the radian measure in the interval $[0, 2\pi]$ associated with it. The point $(-1, 0)$, associated with a radian measure of π, is done for you.

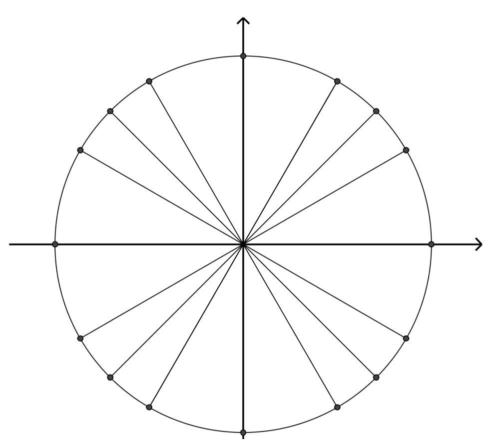


2. Assuming the lines drawn in the diagrams correspond to the "special" angles, label each terminal point with the radian measure in the specified interval of its angle.

a) $-2\pi \leq \theta < 0$



b) $2\pi < \theta \leq 4\pi$

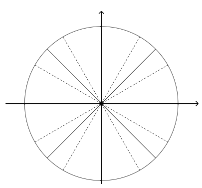


Section 4.2 The Sine and Cosine Functions

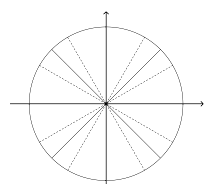
Girt's x-coordinate function that you graphed in TAI 4.1.4 is much more commonly known as the cosine function and her y-coordinate function from TAI 4.1.5 is known as the cosine function. In other words, an input to the cosine or sine function can be thought of as a (signed) distance or angle associated with a point on unit circle while the outputs of the cosine and sine function are, respectively, the point's x- and y-coordinates.

Think About It 4.2.1 Find the value of each expression. Make a dot to show the point on the unit circle associated with the angle of which you are finding the sine or cosine and draw a perpendicular from the dot to the appropriate axis to show whether you are interested in the $x-$ or $y-$coordinate.

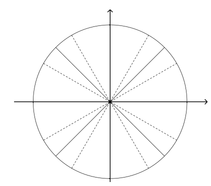
a. $\sin\frac{\pi}{2}$



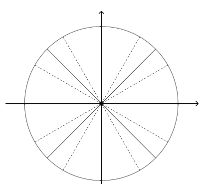
b. $\cos\frac{\pi}{4}$



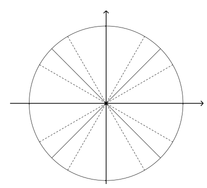
c. $\cos\pi$



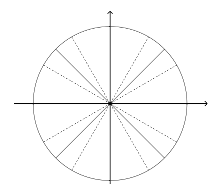
d. $\sin\left(-\frac{\pi}{6}\right)$



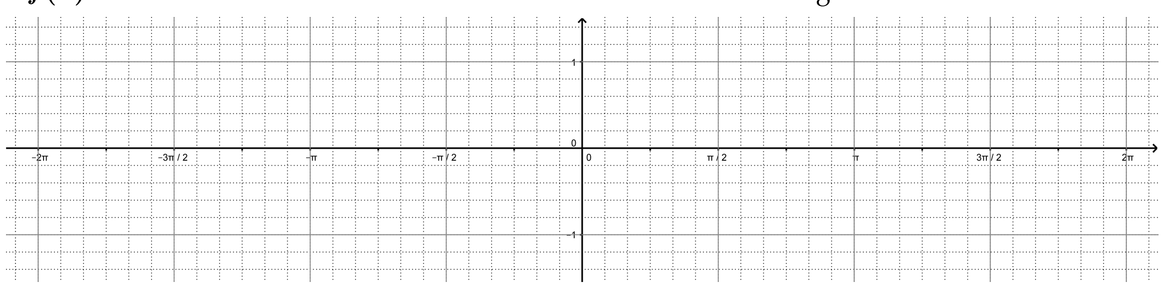
e. $\sin\frac{7\pi}{4}$



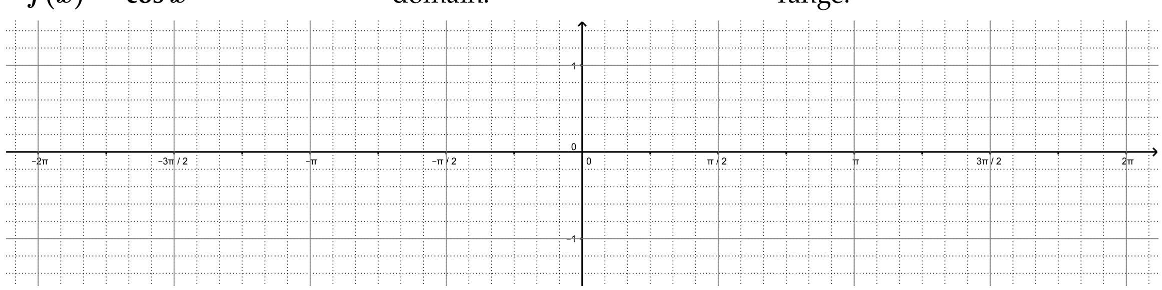
f. $\cos\left(-\frac{4\pi}{3}\right)$



$\boldsymbol{f(x) = \sin x}$ domain: range:



$\boldsymbol{f(x) = \cos x}$ domain: range:

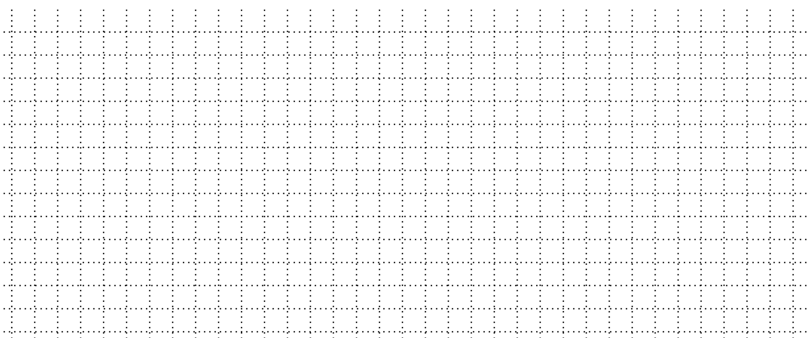


The sine and cosine functions are examples of **periodic** functions. Informally speaking, we might say that in periodic functions, we see the same sequence of outputs for successive equal-width portions of the domain. The width of one of those equal-width portions of the domain is the **period** of the function. Formally, we have the following definition of these terms.

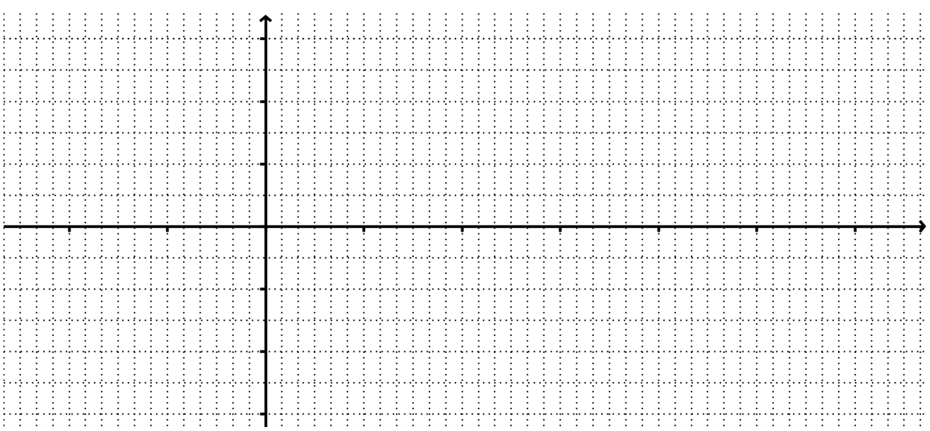
Definition 4.2.1 A function f is said to be **periodic** if and only if there exists a positive constant p such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest possible value of p is called the **period** of f.

Think About It 4.2.2 Use Definition 4.2.1 to explain why the cosine function is a periodic function which has a period of 2π.

Think About It 4.2.3 Sketch the graph of a function with a period of 5.



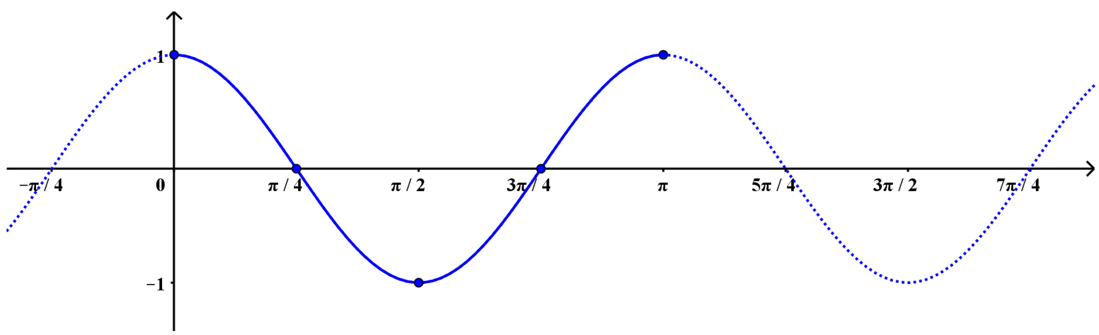
Think About It 4.2.4 Carefully graph the function $f(t) = 20\cos(9t)$.
(Hint: Just draw the cosine graph as you normally would and then figure out how to label the tick marks on the axes.)



Think About It 4.2.5 State the range and the zeros of $g(w) = 5\sin(3w) + 11$.

Example 4.2.1 Solve $\cos(2x) = 0$ by graphing.

Solution To graph the function on the left side of the equation we think about the fact that this is a horizontal compression of the standard cosine function by a factor of 2. Therefore, to graph one period, we begin as we would when graphing $y = \cos x$, by making four equally spaced marks along the x-axis (with each mark corresponding to a boundary between quadrants on a trip around the unit circle). We then graph exactly the same shape we would when graphing the parent function. Instead of labeling the last of our marks 2π, however, we label it π to account for the horizontal compression. This means the first mark is $\frac{\pi}{4}$, the next is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$, and the third is $\frac{3\pi}{4}$.



We see, then, that two solutions to the equation $\cos(2x) = 0$ are $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Of course, these are not all of the solutions, since the domain of our function is $(-\infty, \infty)$. Every time we add (or subtract) the period of the function to one of our solutions we'll get another solution. A standard way of writing this is to say that the solutions are $x = \frac{\pi}{4} + k\pi$ and $x = \frac{3\pi}{4} + k\pi$. By convention, it is implied in this notation that k represents any integer.

It's worth noting that we could specify the solutions even more succinctly in this case by noticing that the two solutions within a period are exactly half a period apart, which suggests writing the solution as $x = \frac{\pi}{4} + \frac{\pi}{2}k$. As a general technique, it works well to look for all solutions within a single period and then add an integer multiple of the period to each. If you care about expressing your final answer as elegantly as possible, however, you'll want to look for any symmetry that allows for a simpler statement of the solution.

Example 4.2.2 Solve $\cos(2x) = 0$ algebraically.

Solution One way to begin is to reduce the clutter in the problem by replacing $2x$ with θ, which results in the equation $\cos\theta = 0$. We know that in one counter-clockwise trip around the unit circle we find two places where the cosine is 0, namely, when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{3\pi}{2}$. Of course, traveling a full rotation from a solution gives us another solution, so we say the solutions to our de-cluttered equation are

$$\theta = \frac{\pi}{2} + 2\pi k \quad \text{and} \quad \theta = \frac{3\pi}{2} + 2\pi k$$

Of course, we were asked to solve for x, not θ, so we now need to substitute $2x$ back in for θ:

$$2x = \frac{\pi}{2} + 2\pi k \quad \text{and} \quad 2x = \frac{3\pi}{2} + 2\pi k$$

The final step is to divide both sides of each equation by 2:

$$x = \frac{\pi}{4} + \pi k \quad \text{and} \quad x = \frac{3\pi}{4} + \pi k$$

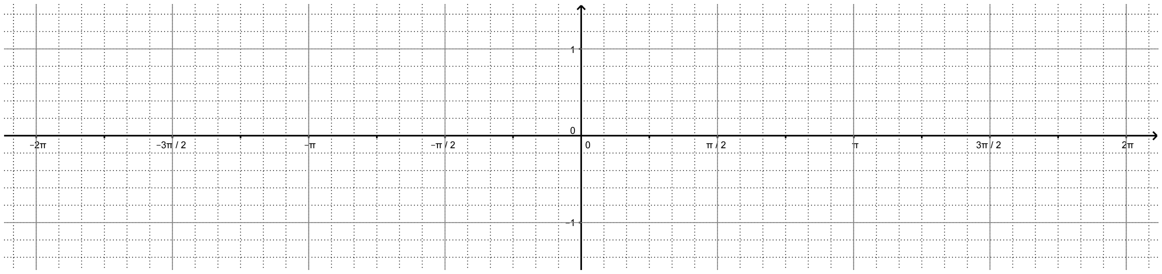
Happily, this solution is the same as that obtained in Example 4.2.1 (and can also be written more elegantly, if you like).

Think About It 4.2.6 Use your knowledge of transformations to predict the where the zeros of $g(x) = \cos\left(2x + \frac{\pi}{3}\right)$ will be. Then solve the equation $\cos\left(2x + \frac{\pi}{3}\right) = 0$ algebraically to check your prediction.

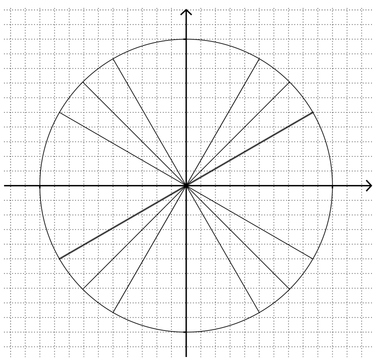
Think About It 4.2.7 If $f_1(x) = \cos x$, find $f_2(x) = f_1(2x)$ and $f_3(x) = f_2\left(x + \frac{\pi}{6}\right)$. Why are you being asked to think about this?

•Solving Equations Involving Sine and Cosine•

Think About It 4.2.8 Explain how you could use the graph of $f(x) = \sin x$ to solve the equation $\sin x = \frac{1}{2}$.



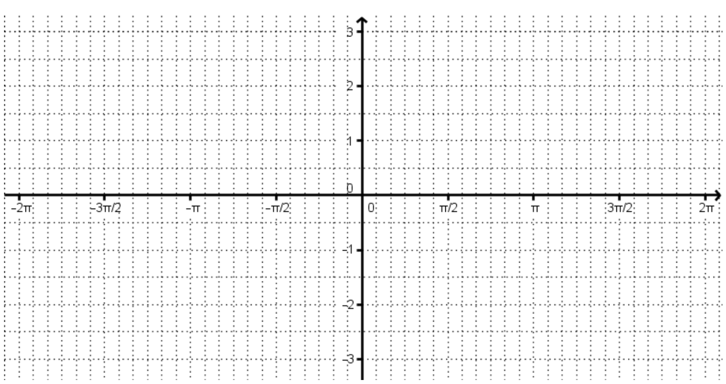
Think About It 4.2.9 Explain how you could use the unit circle to solve the equation $\sin x = \frac{1}{2}$.

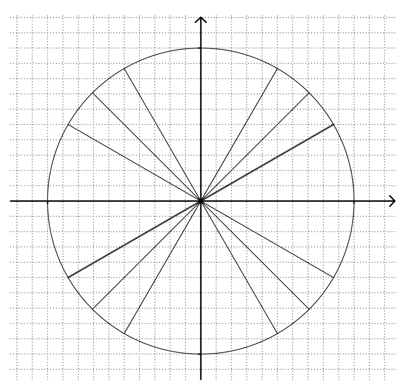


Think About It 4.2.10 Solve the equation $\sin(3x) = \frac{1}{2}$ for values of x on the indicated interval.

a) $[0, 2\pi)$

b) $(-\infty, \infty)$





Think About It 4.2.11 State two different equations for which the solutions are $x = \frac{3\pi}{2} + k\pi$ (for integer values of k). What would be another way of expressing this same set of solutions? And another?

Example 4.2.3 Solve: $2(\cos\theta)^2 + 3(\cos\theta) = -1$

Solution While this problem may look intimidating, we are already quite good at solving a simpler version of the problem. This simpler version magically appears if we just replace $\cos\theta$ with our good friend x:

$$2x^2 + 3x = -1$$

Ahh. A simple quadratic equation which turns out to be solvable by factoring.

$$2x^2 + 3x + 1 = 0$$
$$(2x+1)(x+1) = 0$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = 1$$

Since we had used x as a stand-in for $\cos\theta$ we can now determine what $\cos\theta$ is:

$$\cos\theta = -\frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$

The solution to the first of these equations is

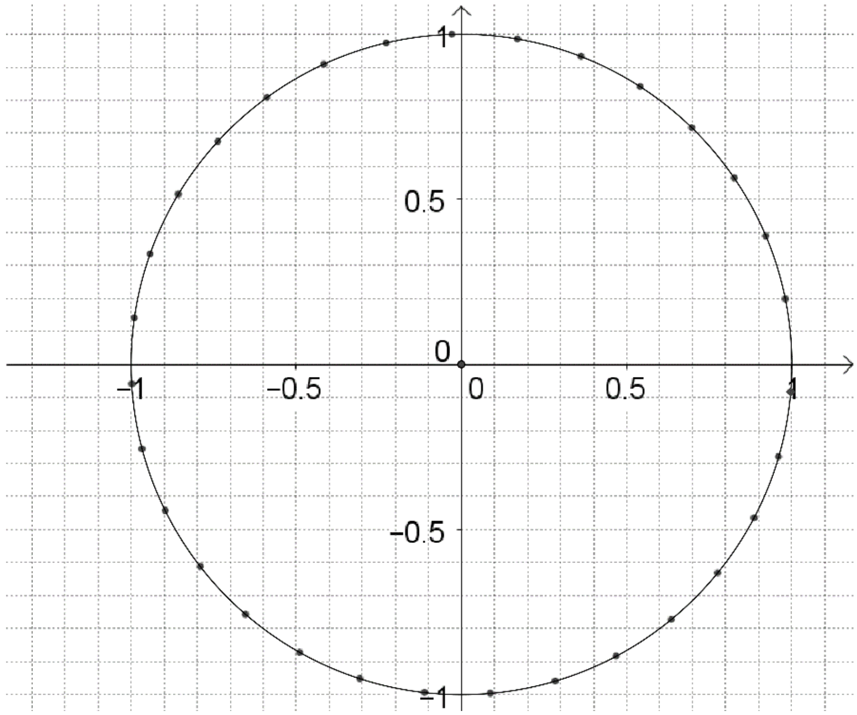
$$\theta = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad \theta = \frac{4\pi}{3} + 2k\pi$$

While the solution to the second is

$$\theta = \pi + 2k\pi$$

Problem Set 4.2

1. In the unit circle diagram below, dots have been placed along the circle at intervals of two-tenths of a unit. Use the diagram to provide careful estimates of the indicated sines and cosines.



a) $\sin 0.2$

b) $\cos 0.4$

c) $\sin \frac{1}{2}$

d) $\sin \frac{\pi}{2}$

e) $\cos \frac{\pi}{2}$

f) $\cos(-1)$

g) $\cos(-\pi)$

h) $\sin 2$

i) $\sin 2\pi$

j) $\cos 3$

k) $\cos\left(-\frac{\pi}{4}\right)$

l) $\cos\left(-\frac{1}{4}\right)$

2. Find all possible values of $\sin\theta$ if $\cos\theta = \frac{3}{10}$. What restriction could be placed on θ so that there would be only one possible value of $\sin\theta$?

3. Use the diagram in problem 1 to provide an estimate of the possible values of θ in problem 2.

4. Find all possible values of $\sin\theta$ if $\cos\theta = -\frac{4}{5}$. What restriction could be placed on θ so that there would be only one possible value of $\sin\theta$?

5. Use the diagram in problem 1 to provide an estimate of the possible values of θ in problem 4.

6. Find all possible values of $\cos\theta$ if $\sin\theta = \frac{7}{9}$. What restriction could be placed on θ so that there would be only one possible value of $\cos\theta$?

7. Use the diagram in problem 1 to provide an estimate of the possible values of θ in problem 6.

8. Find all possible values of $\cos\theta$ if $\sin\theta = -\frac{7}{10}$. What restriction could be placed on θ so that there would be only one possible value of $\cos\theta$?

9. Use the diagram in problem 1 to provide an estimate of the possible values of θ in problem 8.

10. Rewrite the given expression, filling in the blank with $>$, $<$, or $=$. Sketch a diagram to support each answer.

a) $\sin 1$ ______ $\sin 2$

b) $\cos 1$ ______ $\cos 2$

c) $\sin \frac{\pi}{3}$ ______ $\sin 1$

d) $\cos \frac{\pi}{3}$ ______ $\cos 1$

In problems 11 to 14, put the given expressions in order from smallest to largest. Give a brief description and/or draw a picture to indicate how you went about determining the order.

11. $\sin 2, \sin 3, \sin\left(\frac{-3\pi}{2}\right), \sin\left(\frac{-3\pi}{7}\right), \sin\frac{11\pi}{10}$

12. $\cos 1, \cos 2, \cos\left(-\frac{\pi}{12}\right), \cos\frac{\pi}{5}, \cos\frac{7\pi}{8}$

13. $\cos 2, \cos\frac{6\pi}{7}, \cos\frac{3\pi}{2}, \cos 0.1, \cos(-1)$

14. $\sin(-3), \sin 2, \sin\left(-\frac{8\pi}{7}\right), \sin\frac{\pi}{4}, \sin\frac{15\pi}{8}$

In problems 15 to 75, state the exact value of each expression.

15. $\sin 0$

16. $\sin\dfrac{\pi}{2}$

17. $\sin\pi$

18. $\sin\dfrac{3\pi}{2}$

19. $\sin 2\pi$

20. $\cos 0$

21. $\cos\dfrac{\pi}{2}$

22. $\cos\pi$

23. $\cos\dfrac{3\pi}{2}$

24. $\cos 2\pi$

25. $\sin\dfrac{\pi}{6}$

26. $\sin\dfrac{\pi}{4}$

27. $\sin\dfrac{\pi}{3}$

28. $\cos\dfrac{\pi}{6}$

29. $\cos\dfrac{\pi}{4}$

30. $\cos\dfrac{\pi}{3}$

31. $\cos\dfrac{5\pi}{4}$

32. $\cos\dfrac{7\pi}{4}$

33. $\cos\dfrac{5\pi}{3}$

34. $\sin\dfrac{2\pi}{3}$

35. $\cos\dfrac{3\pi}{4}$

36. $\cos\dfrac{11\pi}{6}$

37. $\sin\dfrac{7\pi}{6}$

38. $\sin\dfrac{4\pi}{3}$

39. $\sin\dfrac{5\pi}{3}$

40. $\sin 3\pi$

41. $\cos 3\pi$

42. $\cos\dfrac{7\pi}{6}$

43. $\cos\dfrac{2\pi}{3}$

44. $\cos\dfrac{4\pi}{3}$

45. $\sin\dfrac{3\pi}{4}$

46. $\sin\dfrac{7\pi}{4}$

47. $\sin\dfrac{13\pi}{6}$

48. $\sin\dfrac{23\pi}{6}$

49. $\cos\dfrac{11\pi}{4}$

50. $\sin\dfrac{11\pi}{4}$

51. $\sin 44\pi$

52. $\cos 42\pi$

53. $\sin 42\pi$

54. $\cos 137\pi$

55. $\sin 137\pi$

56. $\cos\left(\dfrac{-\pi}{3}\right)$

57. $\sin\left(\dfrac{-\pi}{3}\right)$

58. $\sin\left(\dfrac{-\pi}{2}\right)$

59. $\cos\left(\dfrac{-11\pi}{6}\right)$

60. $\sin\dfrac{10\pi}{3}$

61. $\sin\dfrac{11\pi}{2}$

62. $\cos\left(\dfrac{-5\pi}{3}\right)$

63. $\cos\left(\dfrac{-4\pi}{3}\right)$

64. $\cos\left(\dfrac{-7\pi}{6}\right)$

65. $\sin\left(\dfrac{-5\pi}{6}\right)$

66. $\cos\left(\dfrac{-7\pi}{4}\right)$

67. $\cos\dfrac{7\pi}{2}$

68. $\sin\dfrac{9\pi}{4}$

69. $\sin\left(\dfrac{-2\pi}{3}\right)$

70. $\cos\left(\dfrac{-5\pi}{4}\right)$

71. $\cos\dfrac{17\pi}{6}$

72. $\sin\left(\dfrac{-3\pi}{2}\right)$

73. $\sin\left(\dfrac{-7\pi}{6}\right)$

74. $\sin\left(\dfrac{-7\pi}{4}\right)$

75. $\sin\dfrac{13\pi}{4}$

In problems 76 to 85, solve the equation and provide a labeled sketch to support your answer.

76. $\sin x = -\dfrac{1}{2}$

77. $\cos x = -\dfrac{\sqrt{2}}{2}$

78. $\sin x = 1$

79. $2\cos x = 1$

80. $\cos 2x = 1$

81. $\cos 3x = -1$

82. $2\sin x = \sqrt{3}$

83. $\dfrac{\sin x}{\cos x} = 1$

84. $\dfrac{\sin x}{\cos x} = 0$

85. $\dfrac{\cos x}{\sin x} = 0$

In problems 86 to 97, solve the equation (without the aid of technology).

86. $\sqrt{3}\sin x\cos x - 2\cos x = 0$

87. $2\sin x\cos x = \sqrt{3}\sin x$

88. $2\sin x\cos x + \cos x = 0$

89. $2\sin^2 x - \sin x - 1 = 0$

90. $2\cos(3x) + 1 = 0$

91. $2\cos(2x) + 1 = 0$

92. $4\cos(7x) = -2$

93. $3\sin(2x) + 6 = 0$

94. $4\cos^2 x = 1$

95. $\sin^2(5x) - 1 = 0$

96. $\cos^2 x = \cos x$

97. $2\sin^2 x = 3\sin x + 2$

---

98. If $f(x) = \cos^2(x+9)$ then find three functions g, h, and j such that $f(x) = g(h(j(x)))$.

99. $\cos t = \frac{4}{5}$ and $0 < t < \frac{\pi}{2}$, what are the values of $\cos(\pi - t)$ and $\cos(t + \frac{\pi}{2})$?

100. Explain using symmetry, the unit circle, and the definition of odd and even functions whether whether $f(\theta) = \sin\theta$ is even, odd, or neither.

101. Explain using symmetry, the unit circle, and the definition of odd and even functions whether whether $f(\theta) = \cos\theta$ is even, odd, or neither.

In problems 102 to 104, show how you can tell from the equation of g whether it is even, odd, or neither. There should be both algebraic notation and words in your explanation.

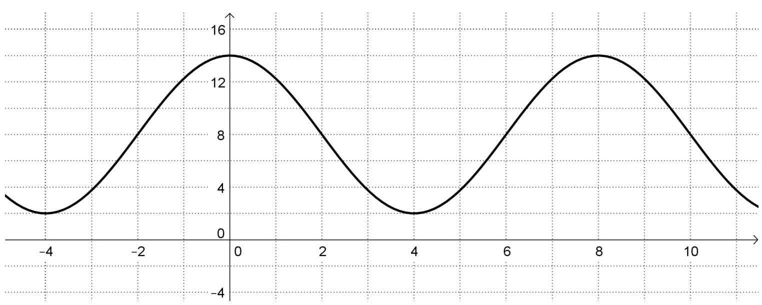
102. $g(x) = x^2\sin x\cos x$

103. $g(x) = x^3\sin x + x\tan x\cos x$

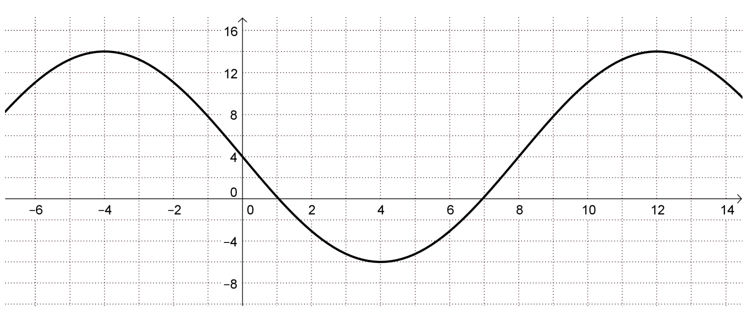
104. $g(x) = x^2\sin x - \sin x\cos x$

In problems 105 to 107, write an equation for the function and state its amplitude and period.

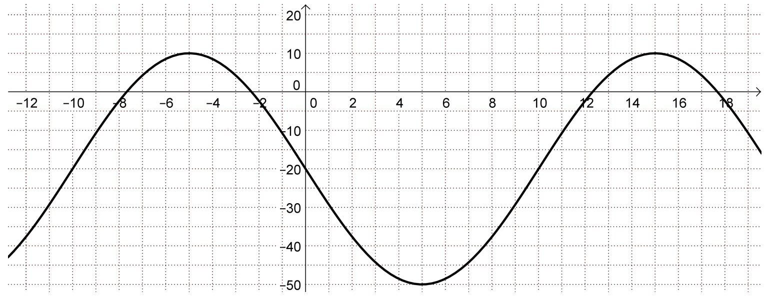
105.



106.



107.



In problems 108 to 123, state the **domain**, **range**, and **period** and, on graph paper, make a careful graph of the function. Label some well-placed tick marks on each axis to indicate scale and carefully place points which represent local extrema as well as points that are on the midline of the function.

108. $f(\theta) = \cos\theta$

109. $f(\theta) = \sin\theta$

110. $f(x) = 3\cos(2x) - 4$

111. $f(x) = -4\sin(\pi x) + 3$

112. $f(x) = 4\sin(6x) + 5$

113. $f(x) = 5\sin(7x) + 8$

114. $g(x) = 3 + \cos(10x)$

115. $f(x) = -\cos\left(\frac{x}{3}\right) + 2$

116. $f(x) = 2 + \sin\left(\frac{x}{8}\right)$

117. $f(x) = 4 - \sin\left(\frac{x}{6}\right)$

118. $f(x) = -10\sin\left(x + \frac{\pi}{4}\right)$

119. $g(x) = 10\cos\left(x - \frac{3\pi}{4}\right)$

120. $g(x) = -8\cos\left(x - \frac{\pi}{3}\right)$

121. $f(x) = 20 - 50\sin\left(2\pi x - \frac{\pi}{4}\right)$

122. $f(x) = 5 - 11\sin\left(2x + \frac{\pi}{4}\right)$

123. $f(x) = 8 - 12\cos\left(\frac{\pi}{4}x - \frac{\pi}{8}\right)$

---

124. Find the average rate of change of the function $P(x) = \cos(x)$ on the interval $[0, \frac{\pi}{3}]$

125. Find the average rate of change of the function $Q(x) = \sin(x)$ on the interval $[\frac{4\pi}{3}, \frac{5\pi}{3}]$

126. Find the average rate of change of the function $R(x) = \sin(x)$ on the interval $[\frac{\pi}{3}, \frac{5\pi}{3}]$

127. Carefully graph the function $f(x) = 12\cos\left(3x - \frac{\pi}{4}\right)$ and use your graph to help you find

a) the *number* of solutions to the equation $f(x) = 10$ on the interval $[0,2\pi]$
b) the exact solutions to the equation $f(x) = 12$ on the interval $[\pi, 3\pi]$
c) all solutions the the equation $f(x) = 12$ (provide a labeled sketch to support your answer)
d) all solutions the the equation $f(x) = 10$ (provide a labeled sketch to support your answer)

128. Use technology to help you find the following if f is the function given in problem 127. Give answers correct to four significant figures and provide labeled sketches to support your answers.

a) all solutions to the equation $f(x) = 11$
b) the solutions to the equation $f(x) = 11$ on the interval $[10\pi, 12\pi]$

129. If $f(x) = 3 + 6\cos\left(5x + \frac{\pi}{4}\right)$, find

a) the range of f
b) the values of x for which f has relative minima
c) the *number* of solutions to the equation $f(x) = 4$ on the interval $[0, 4\pi]$
d) the solutions to the equation $f(x) = 3$ on the interval $\left[-\frac{\pi}{5}, \frac{3\pi}{5}\right]$
e) all solutions to the equation $f(x) = 3$

130. Write an equation meeting the specified conditions where one side is a transformation of the sine or cosine function and the other side is a constant. Provide a labeled sketch to support your answer.

a) has exactly one solution on the interval $[0, 2\pi]$
b) has exactly three solutions on the interval $[0, 2\pi]$
c) has no solutions
d) has exactly two solutions on the interval $[0, 2\pi]$ and the solutions represent angles with terminal points in quadrants I and II.
e) has exactly two solutions on the interval $[0, 2\pi]$ and the solutions represent angles with terminal points in quadrants II and III.
f) has exactly two solutions on the interval $[0, 2\pi]$ and the solutions represent angles with terminal points in quadrants II and IV.
g) has exactly six solutions on the interval $[0, 6\pi]$
h) has exactly six solutions on the interval $[0, 2\pi]$
i) has exactly two solutions on the interval $[0, 6\pi]$
j) has exactly five solutions on the interval $[0, 2\pi]$
k) has exactly two solutions on the interval $[0, 15]$
l) has exactly two solutions on the interval $[5, 15]$

131. The London Eye Ferris wheel is 120 meters in diameter and rotates once every thirty minutes. The center axle of Wheel is 75 meters from the ground and passengers board at the bottom of the wheel. Make a rough yet informative sketch to show the height (in meters) above the ground of a passenger as a function of the time (in minutes) since boarding the ride and then write an equation for your graph.

132. As the tide in a body of water rises and falls over the course of a day, the depth of water can be modeled by a sine or cosine function. On April 30, 2014 high tide on the Delaware River in Burlington, NJ occurred at 4:00 AM with a height of 10 ft and the subsequent low tide occurred at 11:00 AM with a height of 2 ft. Assuming that this pattern continues at least until the next high tide, make a rough yet informative sketch to show the water height (in feet) as a function of the time (in hours) since the 4:00 AM high tide and then write an equation for your graph.

133. On May 7, 2014 low tide on the Delaware River at Penn's Landing occurred at 3:00 PM with a height of 0.6 ft and the subsequent high tide occurred at 8:30 PM with a height of 6.8 ft. Assuming that this pattern is sinusoidal and continues at least until the next high tide , make a rough yet informative sketch to show the water height (in feet) as a function of the time (in hours) since the 3:00 PM low tide and then write a equation for your graph.

In problems 134 to 143, use technology to help you find all solutions to the equation on the indicated interval, giving approximations correct to three places after the decimal.

134. $\sin x = -\frac{1}{3}; \quad [0, 2\pi)$

135. $\sin x = -\frac{1}{3}; \quad [-\pi, \pi]$

136. $\sin x = -\frac{1}{3}; \quad [10\pi, 12\pi]$

137. $\sin x = -\frac{3}{5}; \quad [0, 2\pi)$

138. $\sin x = -\frac{3}{5}; \quad [100\pi, 103\pi]$

139. $3\cos(2x) + 1 = 0; \quad [0, 2\pi)$

140. $1 + 7\sin(2x) = 0; \quad [0, 2\pi)$

141. $1 - 5\cos(16x) = 3; \quad (-\infty, \infty)$

142. $\cos(6\theta) = -\dfrac{2}{3}; \quad (-\infty, \infty)$

143. $\cos^2\left(\dfrac{\theta}{5}\right) = \dfrac{1}{3}; \quad (-\infty, \infty)$

---

144. As you work to answer the following questions, it's recommended that you use a graphing utility to graph the equation and then adjust the window variables as necessary.

a) What is the period of $y = \sin \frac{\pi}{2}x + \sin \frac{\pi}{3}x$?

b) What is the period of $y = \sin \frac{\pi}{2}x + \sin \frac{\pi}{5}x$?

c) What is the period of $y = \sin \frac{\pi}{6}x + \sin \frac{\pi}{4}x$

d) What's the pattern? That is, what is the period of $y = \sin \frac{\pi}{a}x + \sin \frac{\pi}{b}x$, in terms of a and b, if a and b are integers not equal to zero?

e) Generalize the above pattern. What's the period of $y = \sin \frac{\pi}{a_1}x + \sin \frac{\pi}{a_2}x + \cdots + \sin \frac{\pi}{a_n}x$, if all a_i are integers not equal to zero?

Thanks to Sam Shah whose blog post on Inverse Trig Functions inspired problems 1 to 9 and problems 127 to 130.[1]

[1]Shah, Sam J. "Inverse Trig Functions." Continuous Everywhere but Differentiable Nowhere. August 11, 2013. Accessed August 27, 2016. https://samjshah.com/2013/08/11/inverse-trig-functions/.

Section 4.3 Tangent, Cotangent, Secant, and Cosecant Functions

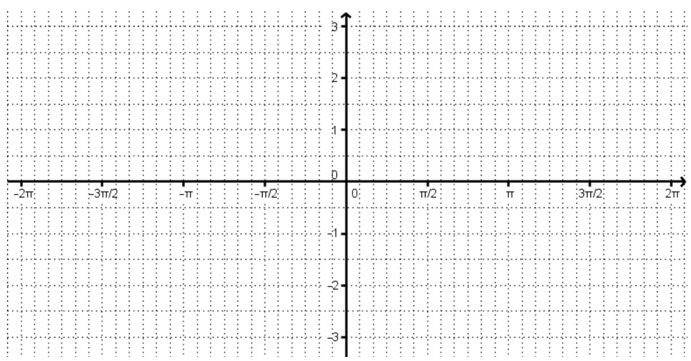
We define four more trigonometric functions in terms of the sine and/or cosine functions:

Definition 4.3.1 $\tan x = \dfrac{\sin x}{\cos x}$ $\cot x = \dfrac{\cos x}{\sin x}$ $\sec x = \dfrac{1}{\cos x}$ $\csc x = \dfrac{1}{\sin x}$

Although these functions are not rational functions (because $\sin x$ and $\cos x$ aren't polynomials), it's worth noting that you can use the same method you used to find the vertical asymptotes and zeros of rational functions to find the vertical asymptotes and zeros of these functions.

Think About It 4.3.1 Use your understanding of the sine and cosine functions to help you graph the other four trig functions. Show the asymptotes and carefully plot any points associated with special inputs where the output is equal to $0, \pm\frac{1}{2}, \pm 1$, or $\pm\, 2$. You may find it useful to begin with the zeros and asymptotes and then think about where the function values will be positive and where they will be negative. In graphing th secant and cosecant functions, consider lightly sketching in the sine or cosine graph first.

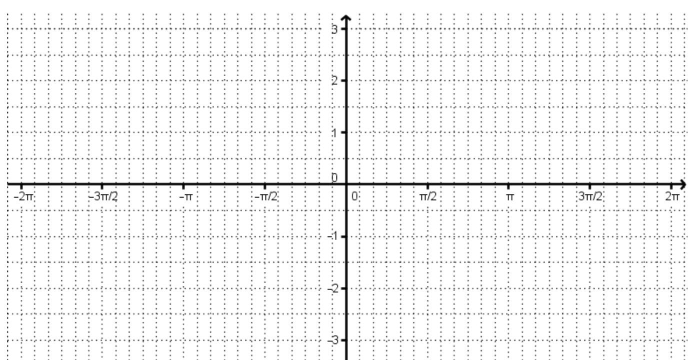
$f(x) = \sec x$



domain:

range:

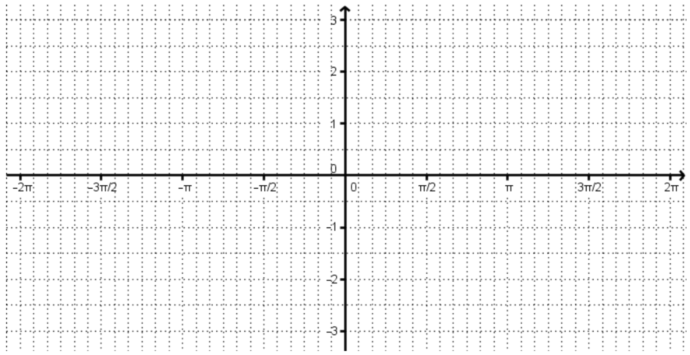
$f(x) = \csc x$



domain:

range:

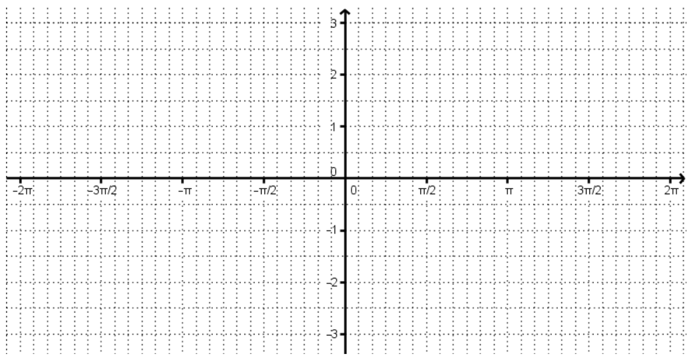
$f(x) = \tan x$



domain:

range:

$f(x) = \cot x$



domain:

range:

Think About It 4.3.2 Use the unit circle and the relationships among the trig functions to help explain why the graphs of all six trig functions are above the x-axis on the interval $\left(0, \frac{\pi}{2}\right)$.

Exploration 4.2

Use technology to help you make a rough sketch of the graph of each of the following equations. Can you use the equations to prove why each graph should look the way it does?

E1. $y = \dfrac{(\sin x)^4 - (\cos x)^4}{(\sin x)^2 - (\cos x)^2}$

E2. $y = (\sec x)^2 - (\tan x)^2$

E3. $y = (\csc x)^2 - (\cot x)^2$

E4. $y = \sec x - (\tan x)(\sin x)$

E5. $y = \dfrac{\cos x}{1 - \sin x} - \sec x$

Problem Set 4.3

In problems 1 to 35, state the exact value of each expression.

1. $\tan 0$

2. $\tan \dfrac{\pi}{6}$

3. $\tan \dfrac{\pi}{4}$

4. $\tan \dfrac{\pi}{3}$

5. $\tan \dfrac{\pi}{2}$

6. $\tan \dfrac{2\pi}{3}$

7. $\csc\left(\dfrac{-3\pi}{4}\right)$

8. $\sec \dfrac{\pi}{6}$

9. $\tan \dfrac{5\pi}{4}$

10. $\sec \dfrac{5\pi}{3}$

11. $\sec\left(\dfrac{-5\pi}{4}\right)$

12. $\cot \dfrac{\pi}{3}$

13. $\cot 2\pi$

14. $\sec \dfrac{\pi}{3}$

15. $\tan \dfrac{3\pi}{2}$

16. $\cot \dfrac{2\pi}{3}$

17. $\tan\left(\dfrac{-7\pi}{6}\right)$

18. $\tan \dfrac{3\pi}{4}$

19. $\tan \pi$

20. $\csc 0$

21. $\tan 2\pi$

22. $\csc(-2\pi)$

23. $\tan \dfrac{7\pi}{4}$

24. $\sec\left(\dfrac{-5\pi}{3}\right)$

25. $\sec(-\pi)$

26. $\csc 2\pi$

27. $\cot \dfrac{\pi}{2}$

28. $\cot \pi$

29. $\csc \dfrac{4\pi}{3}$

30. $\cot\left(\dfrac{-\pi}{3}\right)$

31. $\sec\left(\dfrac{-\pi}{2}\right)$

32. $\tan\left(\dfrac{-11\pi}{6}\right)$

33. $\cot\left(\dfrac{-\pi}{6}\right)$

34. $\sec \dfrac{4\pi}{3}$

35. $\tan\left(\dfrac{-5\pi}{3}\right)$

In problems 36 to 43, let θ be an angle in standard position with its terminal side in the indicated quadrant. Use this information along with the value of the given trig function to find the values of the other five trig functions.

36. $\csc\theta = \frac{13}{5}$, Q II

37. $\sin\theta = \frac{1}{3}$, Q II

38. $\cos\theta = \frac{3}{4}$, Q IV

39. $\cos\theta = -\frac{2}{3}$, Q III

40. $\tan\theta = \frac{2}{3}$, Q III

41. $\cot\theta = -\frac{1}{2}$, Q IV

42. $\tan\theta = -3$, Q IV

43. $\cot\theta = -\sqrt{2}$, Q II

---

44. If $\sec\theta > 0$ and $\cot\theta < 0$, in which quadrant is θ located?

45. The terminal point on a unit circle associated with angle θ has an x-coordinate of $-\frac{2}{3}$ and a positive y-coordinate. Find the six trig functions of θ.

46. If $\tan\theta = \frac{7}{24}$, and $\sin\theta < 0$, give the values of the six trigonometric functions at θ.

47. In what quadrant does the terminal side of the angle θ in problem 46 lie? Use your calculator to find two possible values in degrees of θ.

In problems 48 to 55, make a careful graph, on graph paper, of the function and state its domain, range, period, and the equations of any asymptotes.

48. $f(\theta) = \tan\theta$

49. $f(\theta) = \cot\theta$

50. $f(\theta) = \sec\theta$

51. $f(\theta) = \csc\theta$

52. $f(\theta) = \tan 2\theta$

53. $f(\theta) = \cot\left(\dfrac{\theta}{2}\right)$

54. $f(\theta) = 3 + \sec\theta$

55. $f(\theta) = 1 + 2\csc\theta$

In problems 56 to 57, solve the equation (without the aid of technology).

56. $3\csc^2 x - 6 = 0$

57. $\sec^2 x - \sec x = 2$

---

58. Use technology to help you find all solutions to $(\tan x + 5)(4\sin x - 3) = 0$

59. Use technology to help you find all solutions to $(\tan 4x - 8)(5\sin x + 1) = 0$

Section 4.4 Trigonometric Identites

An immediate consequence of defining $\cos\theta$ and $\sin\theta$ as the x- and y-coordinates respectively of the terminal point on the unit circle associated with angle θ is the fact that $(\sin\theta)^2 + (\cos\theta)^2 = 1$. (Why?) This equation, like any equation true for all values of the input variable, is called an **identity**. This particular identity is one of a set of three identities known as the **Pythagorean Identities**. (Why?)

Think About It 4.4.1 What do you end up with if you divide both sides of $(\sin\theta)^2 + (\cos\theta)^2 = 1$ by $(\cos\theta)^2$ and simplify?

Think About It 4.4.2 What do you end up with if you divide both sides of $(\sin\theta)^2 + (\cos\theta)^2 = 1$ by $(\sin\theta)^2$ and simplify?

It's worth noting that $(\sin\theta)^2 + (\cos\theta)^2 = 1$ is more commonly written as $\sin^2\theta + \cos^2\theta = 1$. For some reason, mathematical notation has developed in such a way that it is an acceptable—and even encouraged—practice to express powers of trig functions by using an exponent immediately after the function name, something which is simply not done with other functions. For instance, it is not acceptable to write $\ln^2 3$ to mean $(\ln 3)^2$.

Pythagorean Identities

Think About It 4.4.3 Does 5 actually equal -5? Consider the proof(?) below. What's going on?

$$\begin{aligned} a &= 5 \\ a^2 &= 25 \\ a^2 - 25 &= 0 \\ (a+5)(a-5) &= 0 \\ \frac{(a+5)(a-5)}{a-5} &= \frac{0}{a-5} \\ a+5 &= 0 \\ a &= -5 \\ \therefore \quad 5 &= -5 \end{aligned}$$

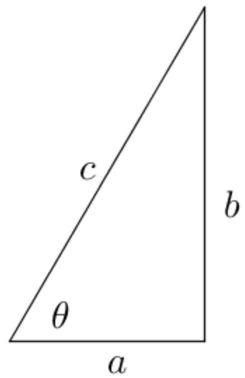
Think About It 4.4.4 Which of the following statements are true? Sketch graphs to support your answers.

$\sin(-x) = -\sin x$ $\tan(-x) = -\tan x$ $\sec(-x) = -\sec x$

$\cos(-x) = -\cos x$ $\cot(-x) = -\cot x$ $\csc(-x) = -\csc x$

Odd/Even Identities

Think About It 4.4.5 Convince yourself that the measure of the unlabeled angle in the triangle below must be $\frac{\pi}{2} - \theta$. Then find the expressions for the six trig functions of θ and of $\frac{\pi}{2} - \theta$ in terms of a, b, and c.



Co-function Identities

Exploration 4.3

If you're being particularly attentive, you may have realized that the work you did in TAI 4.4.5 only proved the co-function identities for $0 < \theta < \frac{\pi}{2}$. Use the odd/even identities together with your understanding of function transformations (or any other approach you can think of) to prove that the co-function identities hold for all possible values of θ.

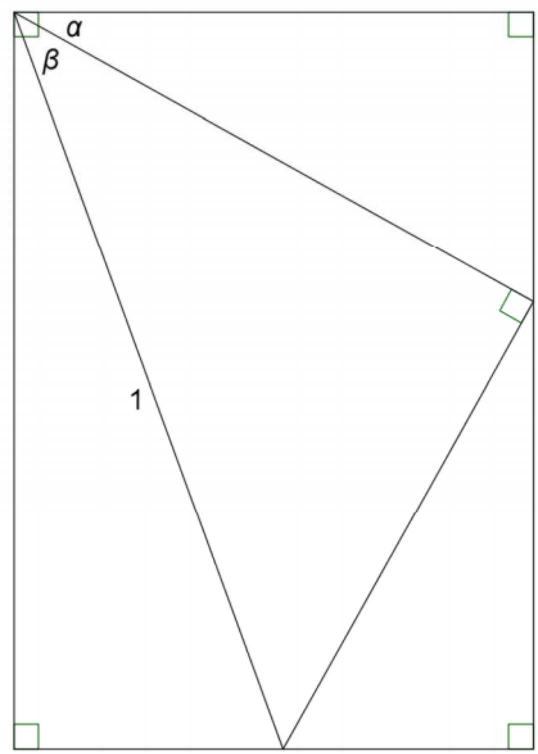
Exploration 4.4

Figure out whatever you can about the graphs of

- $f(x) = 2\sin x \cos x$
- $g(x) = 2\cos^2 x - 1$
- $h(x) = 1 - 2\sin^2 x$
- $j(x) = \cos\frac{5\pi}{12}\cos\frac{\pi}{12} - \sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

Exploration 4.5

The rectangle shown is divided into four right triangles.



E1. One of the unmarked angles in the diagram must have a measure of $\alpha$and one must have a measure of $\alpha + \beta$. Find these angles.

E2. Write expressions involving sin, cos, α, and β to represent the legs of the triangles that make up the sides of the rectangle.

E3. Write an equation stating that the left side of the rectangle is the same length as the right side.

E4. Write an equation stating that the top side of the rectangle is the same length as the bottom side.

You now have identities which tell you how to write $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$, and $\cos\beta$!

You try an online version of this exploration at http://ggbm.at/gNVjYaPy

Think About It 4.4.6 Use an identity you discovered in Exploration 4.5 to simplify the expression $\sin\dfrac{2\pi}{5}\cos\dfrac{7\pi}{20} + \cos\dfrac{2\pi}{5}\sin\dfrac{7\pi}{20}$

Think About It 4.4.7 Replace β by $-\beta$ in the identities you discovered in Exploration 4.5 and then use odd/even identities to find out how to write $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$, and $\cos\beta$.

Angle Sum and Difference Identities

Exploration 4.6

The work you did in Exploration 4.5 actually only proved the angle sum identities for the case where α, β, and $\alpha+\beta$ are all between 0 and $\frac{\pi}{2}$. One way to prove them for all angle combinations is to begin by considering the interactive diagram at http://ggbm.at/zdru9qN4. Finding the lengths of the two versions of the dotted segment in this diagram and doing some algebra can lead you to the cosine of a difference formula. From there, identities proven earlier in this section can help you derive the other three angle sum and difference identities.

Think About It 4.4.8 Replace β by α in the angle sum identities, do a little algebra, and use one or more identities you've proven already to figure out why the graphs of f, g, and h in Exploration 4.4 look the way they do.

Double Angle Identities

Problem Set 4.4

In problems 1 to 3, use identities to simplify the expression.

1. $\cos y(\tan y - \sec y)$

2. $\sin^3\theta + \sin\theta\cos^2\theta$

3. $\sin^2\theta - \cot^2\theta + \cos^2\theta + \sec^2\theta - \tan^2\theta + \csc^2\theta$

In problems 4 to 21, prove that the given equation is true for all values of the variables for which both sides are defined.

4. $\dfrac{\sin A}{\cos A} = \tan A$

5. $\sin B + \cos B \cot B = \csc B$

6. $\dfrac{\sec\alpha\sin^2\alpha}{\sec^2\alpha - 1} = \cos\alpha$

7. $\cos^2\theta - \sin^2\theta = \cos^4\theta - \sin^4\theta$

8. $2\cos^2\beta - 1 = 1 - 2\sin^2\beta$

9. $\dfrac{1+\cot\phi}{\csc\phi} - \cos\phi = \sin\phi$

10. $\cos(x) - \sin(-x) = \cos x + \sin x$

11. $\dfrac{\cos^2\alpha - \sin^2(-\alpha)}{\sin\alpha - \cos(-\alpha)} = \sin(-\alpha) - \cos\alpha$

12. $\dfrac{\cot\theta\left(1+\tan^2\theta\right)}{\csc\theta\tan\theta} = \csc\theta$

13. $\dfrac{1+\cot\alpha}{1+\tan\alpha} = \cot\alpha$

14. $\dfrac{\sec(-\beta) - \cos\beta}{\tan\beta} = \sin\beta$

15. $\tan\alpha + \dfrac{\cos(-\alpha)}{1-\sin(-\alpha)} = \sec\alpha$

16. $\dfrac{\tan\theta + \cot\theta}{\sec(-\theta)} = \csc\theta$

17. $\dfrac{\cos\lambda}{1+\sin\lambda} = \sec\lambda - \tan\lambda$

18. $(1-\cos\phi)(1+\sec\phi) = \sin\phi\tan\phi$

19. $\dfrac{2\tan\beta}{1+\tan^2\beta} = \sin 2\beta$

20. $\sec^2\lambda + \tan\lambda\sec\lambda = \dfrac{1}{1-\sin\lambda}$

21. $\tan\theta + \cot\theta = \csc\theta\sec\theta$

In problems 22 to 30, use identities to help you find the exact value of the expression.

22. $1 - 2\sin^2\left(\dfrac{3\pi}{8}\right)$

23. $2\cos^2\dfrac{5\pi}{12} - 1$

24. $\cos^2\left(\dfrac{5\pi}{12}\right) - \sin^2\left(\dfrac{5\pi}{12}\right)$

25. $\tan^2\left(\dfrac{3\pi}{8}\right) - \sec^2\left(\dfrac{3\pi}{8}\right)$

26. $\sin\dfrac{\pi}{12}\cos\dfrac{\pi}{12}$

27. $\sin\left(\dfrac{7\pi}{8}\right)\cos\left(\dfrac{7\pi}{8}\right) - \sin\left(-\dfrac{7\pi}{8}\right)\cos\left(-\dfrac{7\pi}{8}\right)$

28. $\cos^2\dfrac{\pi}{12} - \sin^2\dfrac{\pi}{12}$

29. $\cos\dfrac{10\pi}{7}\cos\dfrac{5\pi}{21} - \sin\dfrac{10\pi}{7}\sin\dfrac{5\pi}{21}$

30. $\sin\dfrac{2\pi}{5}\cos\dfrac{14\pi}{15} + \sin\dfrac{14\pi}{15}\cos\dfrac{2\pi}{5}$

---

31. Use trigonometric sum/difference formulas to evaluate the expression without a calculator.

a) $\sin \dfrac{\pi}{12}$

b) $\cos \dfrac{5\pi}{12}$

32. The expression $\sin\left(\frac{3\pi}{2} - x\right)$ is equivalent to one of the following: $\sin x$, $-\sin x$, $\cos x$, or $-\cos x$. Show how to use an angle sum formula to help prove which it is.

33. The expression $\cos\left(\frac{2\pi}{3} - x\right)$ can be rewritten in the form $a \sin x + b \cos x$. Show how to use an angle sum identity to help determine a and b.

34. Suppose obtuse angle B is such that $\tan B = -\frac{5}{12}$. Evaluate $\sin 2B$ and $\cos 2B$.

35. If $\tan\theta = -\frac{3}{4}$ and $\sin\theta > 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

36. If $\tan\theta = -\frac{12}{5}$ and $\cos\theta > 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

37. If $2\tan^2\theta - 5\sec\theta = 10$, evaluate $\sec\theta$.

In problems 38 to 41, find the solution(s) to the equation on the indicated interval without the aid of technology. Check by graphing with the aid of technology.

38. $2\cos^4\theta - 3\cos^2\theta + 1 = 0; \quad 0 \le \theta \le 2\pi$

39. $6\sin^2 x - 15\sin x = -6; \quad 0 \le x < 2\pi$

40. $\sin^2 x - \cos^2 x = 0; \quad 0 \le x < 2\pi$

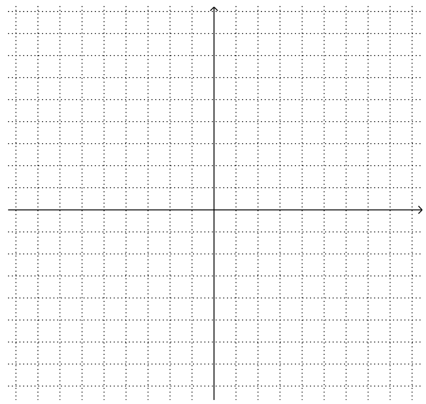
41. $\sin 2x = \sqrt{2}\cos x; \quad 0 \le x \le \pi$

Section 4.5 Inverse Trigonometric Functions

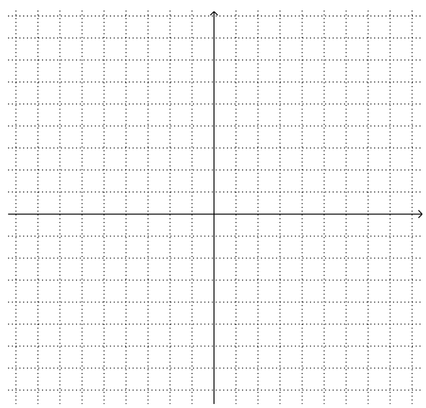
Exploration 4.7

Make a graph which reverses the x- and y-coordinates on the graphs of each of the functions given. Then come up with a way to restrict the range of the graph you've drawn so that the range of your graph remains the same as the domain of the given function and your graph becomes the graph of a function.

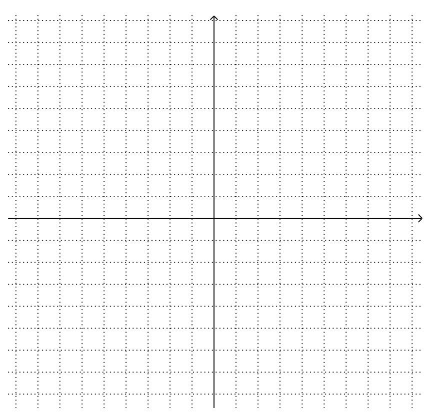
reverse the x- and y-coordinates of $f(x) = \sin x$



reverse the x- and y-coordinates of $f(x) = \cos x$



reverse the x- and y-coordinates of $f(x) = \tan x$



Think About It 4.5.1 State the solution(s) to each equation:

$\sin^{-1}\frac{1}{2} = x$

$\sin x = \frac{1}{2}$

$\sin x = \frac{1}{2}$, where $-\pi \leq x \leq 4\pi$

Primitive Pythagorean Triples with $c < 100$:

$(3,4,5)$	$(5,12,13)$	$(7,24,25)$	$(8,15,17)$
$(9,40,41)$	$(11,60,61)$	$(12,35,37)$	$(13,84,85)$
$(16,63,65)$	$(20,21,29)$	$(28,45,53)$	$(33,56,65)$
$(36,77,85)$	$(39,80,89)$	$(48,55,73)$	$(65,72,97)$

Problem Set 4.5

1. Explain why $\sin^{-1}(-1) \neq \frac{3\pi}{2}$ even though $\sin\frac{3\pi}{2} = -1$.
2. Explain why $\cos^{-1}(-1) \neq -\pi$ even though $\cos(-\pi) = -1$.
3. Explain why $\tan^{-1}(-1) \neq \frac{3\pi}{4}$ even though $\tan\frac{3\pi}{4} = -1$.
4. If $\sin^{-1}(.6) = \theta$, and θ is in quadrant I, find the value of $\sin(\theta)$ and the values for the other five trigonometric functions.

In problems 5 to 31, find the exact value of the expression.

5. $\cos^{-1} 0$
6. $\sin^{-1}\frac{1}{2}$
7. $\tan^{-1} 1$
8. $\tan^{-1}\left(-\sqrt{3}\right)$
9. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
10. $\sin^{-1}(-1)$
11. $\cos^{-1}(-1)$
12. $\tan^{-1}(-1)$
13. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
14. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
15. $\sin^{-1}\left(\frac{\pi}{2}\right)$
16. $\cos^{-1}\left(-\frac{1}{2}\right)$
17. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$
18. $\sin^{-1}\left(-\frac{1}{2}\right)$
19. $\cos^{-1}\left(\frac{\pi}{3}\right)$
20. $\sin\left(\tan^{-1}\sqrt{3}\right)$
21. $\cos\left(\sin^{-1}\frac{40}{41}\right)$
22. $\sin\left(\cos^{-1}\frac{40}{41}\right)$
23. $\csc\left(\cos^{-1}\frac{33}{65}\right)$
24. $\csc\left(\cot^{-1}\frac{40}{9}\right)$
25. $\sec\left(\sin^{-1} 0.5\right)$
26. $\cot\left(\tan^{-1} 1\right)$
27. $\tan\left(\cot^{-1} 4\right)$
28. $\cos\left(\sec^{-1} 2\right)$
29. $\csc\left(\sin^{-1}\left(\cos\left(\tan^{-1}\sqrt{3}\right)\right)\right)$
30. $\sec(\sec^{-1}(332))$
31. $\sin\left(\tan^{-1}\frac{12}{5} + \arctan\frac{3}{4}\right)$

In problems 32 to 40, make a careful graph, on graph paper, of the function and state its domain, range, and the equations of any asymptotes.

32. $f(x) = \sin^{-1} x$

33. $g(x) = \arccos x$

34. $h(x) = \arctan x$

35. $f(x) = \cos^{-1}(5x)$

36. $g(x) = 6\tan^{-1} x$

37. $f(x) = 4\cos^{-1} x$

38. $g(x) = \sin^{-1}\left(\frac{x}{3}\right)$

39. $f(x) = 8\sin^{-1} x$

40. $g(x) = \tan^{-1}\left(\frac{x}{4}\right)$

Section 4.6 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

Problem Set 4.6

1. Find the exact value of $\cos \dfrac{61\pi}{6}$ and of $\csc\left(-\dfrac{5\pi}{4}\right)$

2. State the period of $f(x) = 4 + 2\sin(3x)$ and sketch two full periods of its graph. Label several tick marks on each axis to indicate scale and carefully plot points which represent local extrema as well as points that are on the midline of the function.

3. Carefully graph two full periods of the function $f(x) = 2 - 5\sin\left(\frac{\pi}{4}x - \frac{5\pi}{4}\right)$. Draw the x- and y-axes and label some well-placed tick marks on each to indicate scale. Carefully place points which represent local extrema as well as points that are on the midline of the function.

4. Write an equation of the form $a + b\sin(cx) = d$, where none of a, b, or c are 0, which will have the indicated number of solutions. Provide justification for your answers.

a) no solution
b) exactly one solution on the interval $[0, 2\pi]$
c) exactly ten solutions on the interval $[0, 2\pi]$

5. Arrange the expressions in order from least to greatest. Support your answer by annotating a unit circle.

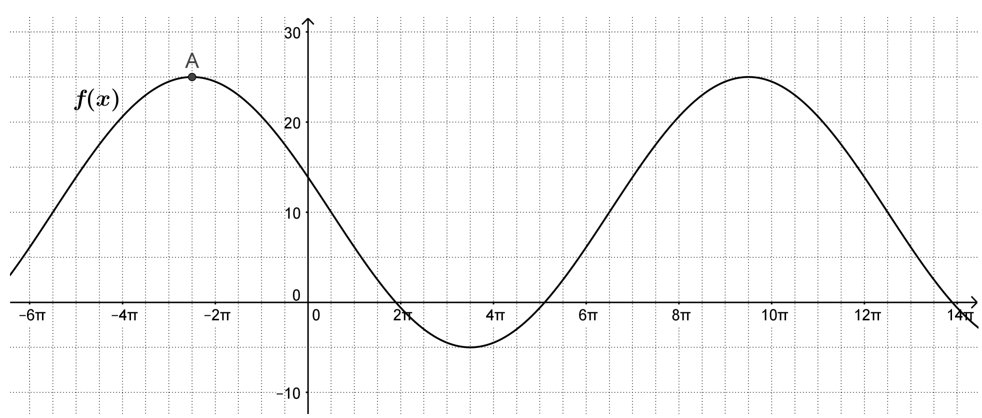
$\sin 1$ $\quad$ $\sin(-3)$ $\quad$ $\sin\left(-\frac{5\pi}{11}\right)$ $\quad$ $\sin\left(\frac{6\pi}{5}\right)$ $\quad$ $\sin\left(\frac{\pi}{10}\right)$

6. The terminal point for angle θ (in standard position) is in the fourth quadrant and $\sec\theta = \dfrac{3}{2}$. Find the values of the other five trigonometric functions of θ.

7. The terminal point for angle θ (in standard position) is in the third quadrant and $\cot\theta = \dfrac{1}{3}$. Find the values of the other five trigonometric functions of θ.

8. Sketch the graph of the function $f(x) = 5 + \sec\left(\frac{x}{3}\right)$, and state its domain, range, and period.

9. The graph of $f(x) = a + b\sin(cx + d)$, which passes through point A is shown.



a) Find the coordinates of point A and the period of f.
b) Find the values of a, b, c, and d.
c) Find the *number* of solutions to the equation $f(x) = 15$ on the interval $[0, 60\pi]$ and a brief explanation of why this is the number of solutions
d) Find all solutions to the equation $f(x) = 10$.
e) Find the solutions to the equation $f(x) = 25$ on the interval $[12\pi, 60\pi]$

10. The first low tide on the East River at the Brooklyn Bridge on March 24, 2016 was predicted to occur at at 4:00 AM with a height of 0.2 ft and the subsequent high tide was predicted to occur at 10:15 AM with a height of 4.6 ft. Assuming that this pattern continues at least until the next low tide, make a rough yet informative sketch to show the predicted water height (in feet) as a function of the time (in hours) since the 4:00 AM low tide and then write an equation for your graph.

11. Find the exact value of each expression.

a) $\arcsin\left(\frac{1}{2}\right)$ b) $\arccos\left(-\frac{1}{2}\right)$ c) $\arctan\left(\frac{\sqrt{3}}{3}\right)$ d) $\arccos\left(\sqrt{2}\right)$

12. Graph $g(x) = 12\cos^{-1}(10x)$, carefully plotting at least five key points, labeling tick marks on the axes to indicate scale, and taking care to show the general shape of the graph.

13. Use one or more identities to help you write each of the following as a single trigonometric function, a constant, or a constant times a single trigonometric function.

a) $\cos\left(\frac{\pi}{2} - x\right)\sec(-x)$ b) $\dfrac{\tan^2 x - \sec^2 x}{\cot x}$

14. Solve: $4\cos^2\phi - 3 = 0$

15. Solve: $1 - 2\sin^2(5x) = 0$

16. Solve: $\sin(2x) - \cos x = 0$

17. Consider the function $f(x) = 2\cos^2 x + \sin x$.

a) Find all solutions to the equation $f(x) = 1$ without the aid of technology.

b) Use a calculator to help you find the solutions to the equation $f(x) = 1$ on $[0, 2\pi)$.

18. Show how you can tell from the equation $g(\theta) = \theta^3 \cdot \sin^2\theta \cdot \tan\theta$ whether g is even, odd, or neither. Include algebraic work and the definition of either an odd or even function or, if necessary, both.

19. Prove: $\sec\alpha \cdot \dfrac{\cos^2\alpha - \sin^2\alpha}{\cos(-\alpha) + \sin(-\alpha)} = 1 + \tan\alpha$

20. Prove: $\dfrac{\sin\beta}{1 - \cos\beta} - \sec\left(\frac{\pi}{2} - \beta\right) = \cot\beta$

21. Evaluate: $1 - 2\sin^2\left(-\frac{\pi}{12}\right)$

22. Evaluate: $\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{\pi}{5}\right) + \sin\left(\frac{8\pi}{15}\right)\sin\left(\frac{\pi}{5}\right)$

23. If θ is in Quadrant IV and $\cos\theta = \frac{2}{3}$, find $\tan 2\theta$.

5 Trigonometric Topics

Section 5.1 The Law of Sines and The Law of Cosines

In any $\triangle ABC$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ (**Law of Sines**)

$$c^2 = a^2 + b^2 - 2ab\cos C$$ (**Law of Cosines**)

Think About It 5.1.1 Given these laws, how many sides and/or angles is it necessary to know in a triangle in order to be able to figure out all the other sides and/or angles?

(Beware SSA!)

Problem Set 5.1

1. Derive the Law of Sines. One approach is to consider $\triangle ABC$ where you know two sides and the included angle. If the angle you know is A write an expression for the area of the triangle in terms of A, b, and c. Write another expression for the area assuming that the angle you know is B. And yet another assuming the angle you know is C. After you've got these three expressions, you're almost there!

2. Derive the Law of Cosines. One approach is to place $\triangle ABC$ in a Cartesian coordinate system with vertex C at the origin and one side running along the positive x-axis. Drop a perpendicular to the axis from the third vertex and find the coordinates of A and B in terms of a, b, and C. Then think about how the distance formula might be helpful.

3. Find the measure of $\angle F$ in $\triangle FOX$ to the nearest hundredth of a degree if $f = 7$, $o = 3$, and $x = 6$.

4. Draw $\triangle DEF$ where $d = 9$ in, $e = 10$ in, and $f = 11$ in. Find the measure of $\angle D$ to the nearest tenth of a degree.

5. Draw $\triangle ABC$ where $\angle A = 50°$, $\angle B = 60°$, and $c = 7$ m. Find the length of b, rounded to the nearest tenth. Remember units!

6. Consider $\triangle ABC$ in which $a = 6$, $b = 12$, and $c = 7$.

a) Sketch $\triangle ABC$, labeling parts appropriately.
b) Find the measures of the angles of $\triangle ABC$.
c) Find the area of $\triangle ABC$.

7. Find the area of $\triangle FTS$ if $\angle F = 57°$, $f = 32$m, and $\angle S = 103°$.

8. Solve triangle $\triangle FEI$ if $E = 48°$, $f = 8$, and $e = 10$.

9. Solve triangle $\triangle HNL$ if $h = 29$, $n = 89$, and $l = 66$.

10. Find the area of triangle $\triangle FTS$ if $f = 13$, $t = 29$, and $T = 93°$.

11. Quadrilateral $TCBY$ is such that $TC = 4$, $CB = 8$, $BY = 11$, diagonal $TB = 10$, and $\angle Y = 70°$.

a) Find the measure of $\angle BTY$.
b) Find the measure of $\angle C$.
c) Use the previous work to assist you in finding the area of the quadrilateral $TCBY$.

12. In quadrilateral $ABCD$, $\angle B$ is a right angle, $AB = 9$, $BC = 40$, $CD = 49$, and $\angle C = 102°$. Find the area of the quadrilateral.

13. Two planes leave an airport at the same time. One flies at a constant speed of 640 mi/hr in the direction $65°$ east of due north and the other flies at a constant speed of 820 mi/hr in the direction $24°$ east of due north. To the nearest tenth of a mile, how far apart are the planes after 15 minutes.

14. In $\triangle HEN$, $\angle E = 15°$, $e = 11$, and $n = 20$. There are two triangles that have these measurements. Sketch these triangles and find h, $\angle H$, and $\angle N$ in each. Be sure that in your sketches, obtuse angles are clearly obtuse and acute angles are clearly acute.

15. Find the area of parallelogram $ABCD$ if $\angle A = 60°$, $AB = 5$, and $AD = 8$.

16. Juliette wanders 2 meters from her catfood bowl to a piece of scrap paper on a bearing of 220°. Once there, she then becomes distracted by something shiny 4 meters away and runs toward it on a bearing of 330°. After reaching the shiny object, she realizes that she's hungry again and needs to return to her food bowl.

a) Draw Juliette's situation as described above. Clearly indicate each part of the triangle.
b) What is the distance between Juliette and the food bowl at the time that Juliette has reached the shiny object?
c) What is the bearing that Juliette should use to move from the shiny object to her food bowl?

17. A shiny object is on the side of a river from two people, Bob and Jay, who are standing on the opposite side of that river. They know they're 20 meters away from each other, and that the river is straight and of the same width as far as they can see. When Bob looks at the object he makes an angle of 72° from the side of the river. When Jay looks at the object he makes an angle of 75° from the side of the river. Draw a picture of the situation and determine the width of the river.

18. The sides of a triangle are $x - 2$, x, and $x + 2$ and the largest angle is 120°. Find x. What's the area of the triangle?

19. Cailet is on a bearing of 150° from her house to the bank (she uses a jetpack). After flying the 12 miles to the bank, she then travels 20 miles on a bearing of 260° from the bank to the store.

a) What is the distance from the store to Cailet's house?
b) What is the bearing from the store to Cailet's house?

20. Art is fishing in a large lake. He sees a buoy floating about 14 meters away, on a bearing of 80° from Art. He also sees another fishing boat, about 18 meters away, on a bearing of 280°.

a) What is the distance from the other fishing boat to the buoy?
b) What is the bearing from the other fishing boat to the buoy?

21. Bigfoot has been sighted in a forest by two different rangers. Ranger Al observes that Bigfoot is about 35 yards away from him, and that the bearing to Bigfoot is 104°. Ranger Bob measures that Bigfoot is 41 yards away, and that the bearing to Bigfoot is 40°.

a) What is the distance between the two rangers?
b) What is the bearing from Ranger Al to Ranger Bob?

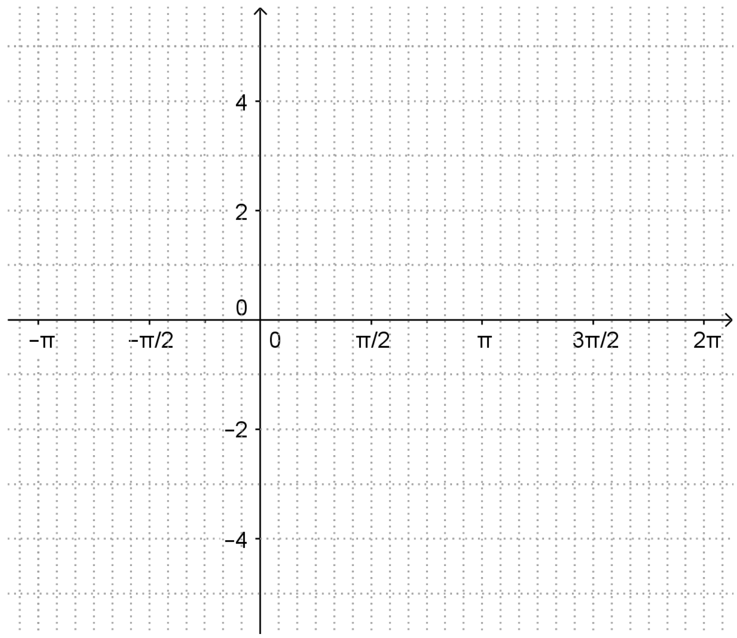
22. A mouse is looking for cheese. There's some cheddar on a bearing of 225° from the mouse, about 4 feet away. There's also some swiss cheese about 5 feet away, on a bearing of 16° from the mouse.

a) What is the distance between the two cheeses?
b) What is the bearing from the cheddar to the swiss?

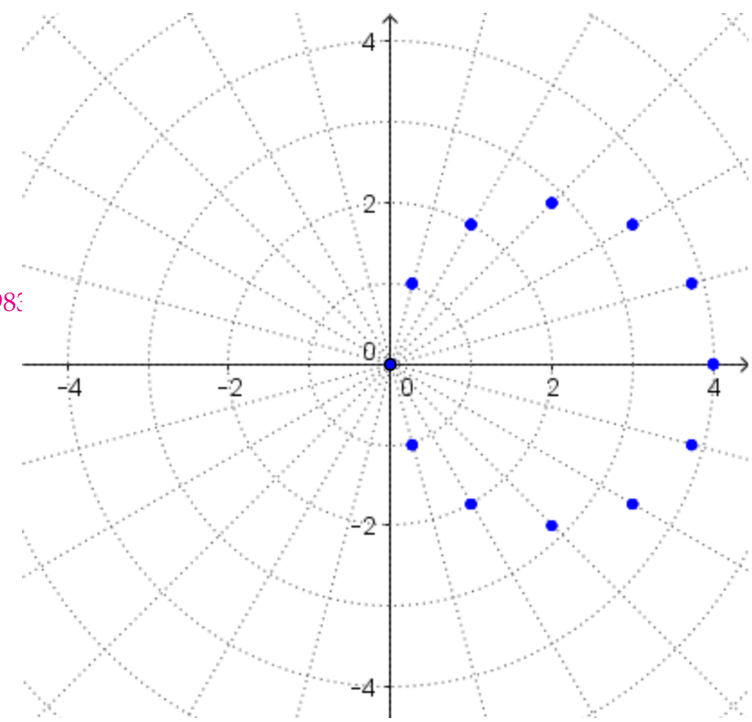
Section 5.2 Polar Coordinates and Graphs

Exploration 5.1

Make a careful graph of the function $r(\theta) = 4\cos\theta$



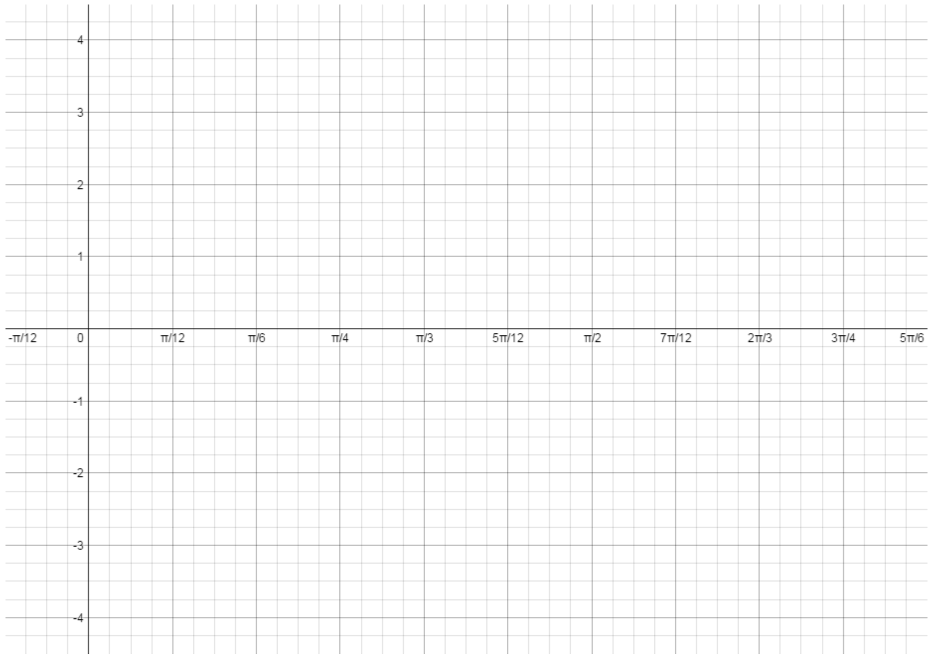
Go to tube.geogebra.org/m/121698: and experiment. Can you figure out how the points in the graph at right are generated from the function you graphed above?

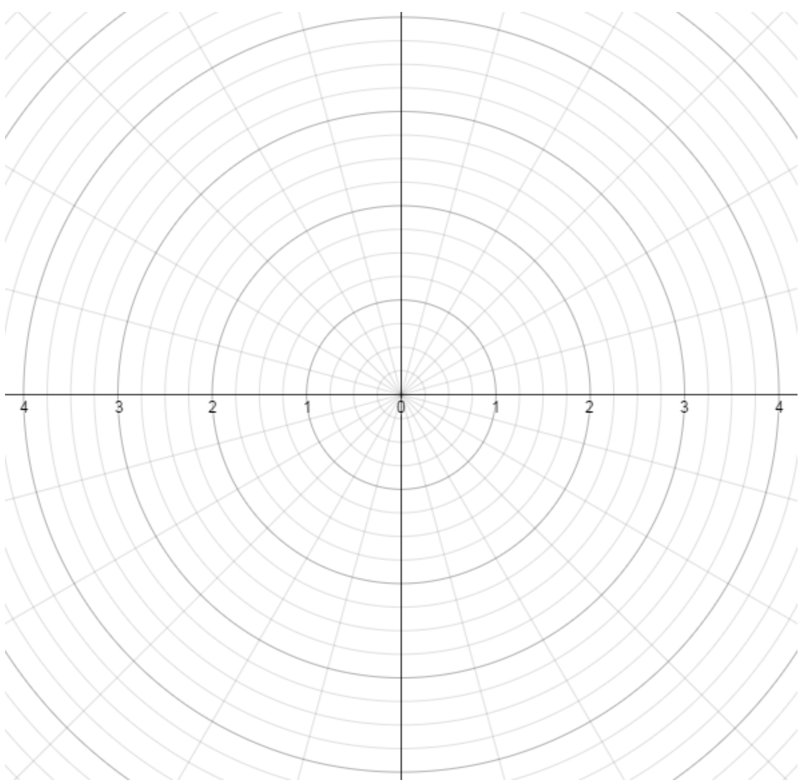


Exploration 5.2

Make a rectangular graph and a polar graph for the equation

$$r(\theta) = 4\sin(3\theta)$$

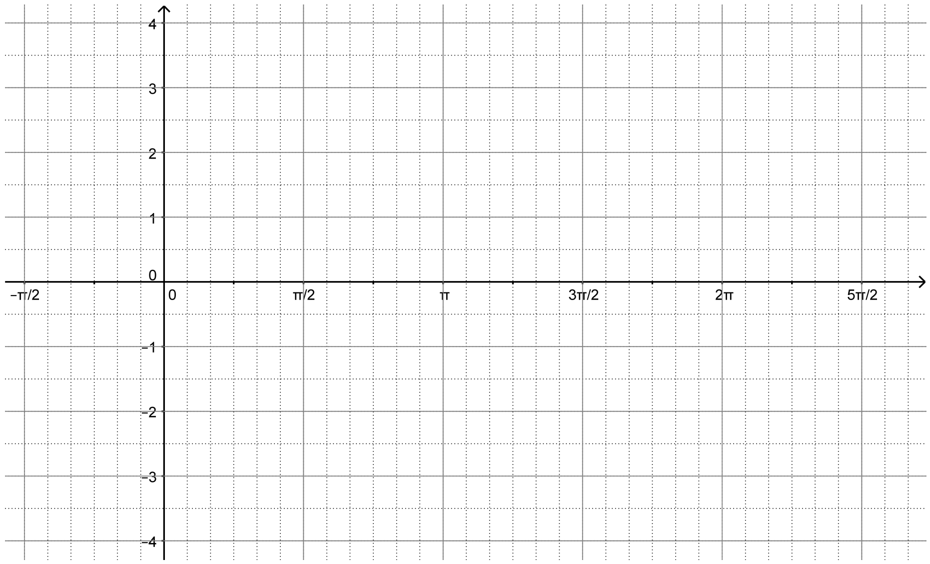


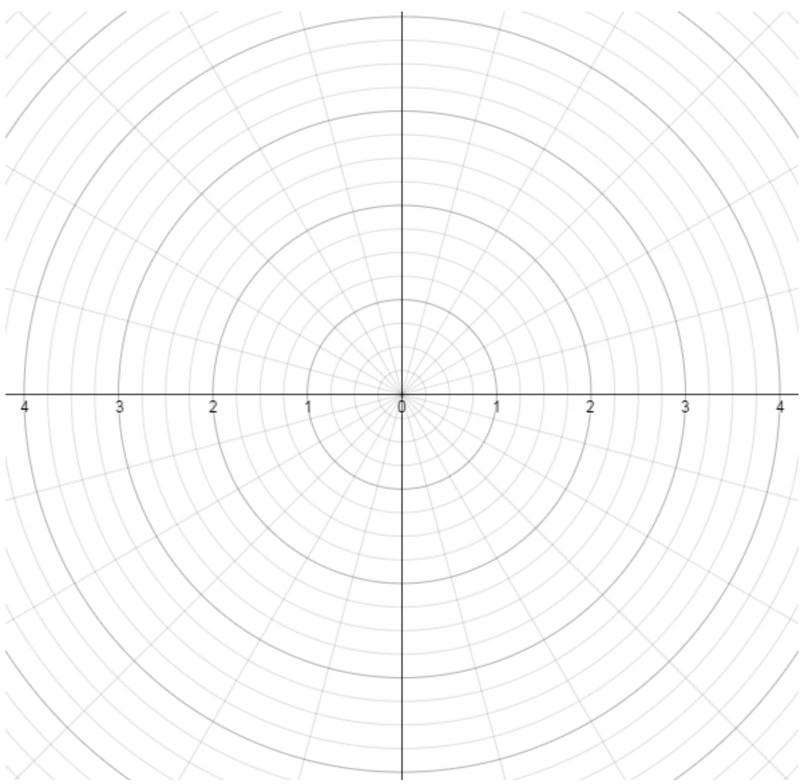


Exploration 5.3

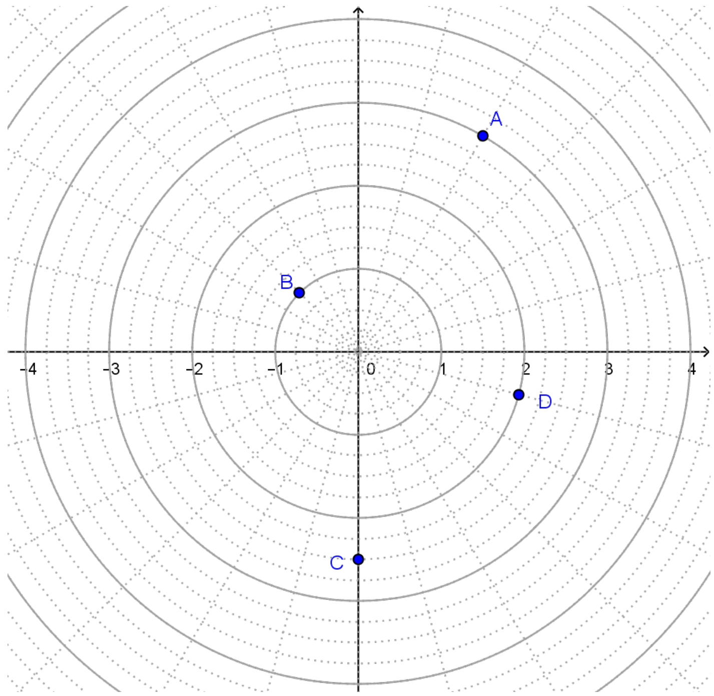
Make a rectangular graph and a polar graph for the equation

$$r(\theta) = 1 - 2\cos\theta$$





Problem Set 5.2



1. Give two sets of polar (r, θ) coordinates for each labeled point in the diagram at right. One must have a positive r and one must have a negative r.
2. Plot one more point somewhere in the third quadrant and give two sets of polar coordinates for it.
3. Make a rectangular graph and a polar graph of the equation $r(\theta) = 3\sin\theta$
4. Make a rectangular graph and a polar graph of the equation $r(\theta) = 4\cos\left(\frac{\theta}{2}\right)$

In problems 5 to 9, graph the given polar point on polar graph paper.

5. $\left(1, -\frac{\pi}{4}\right)$ **6.** $\left(2.5, \frac{\pi}{6}\right)$ **7.** $\left(-\frac{1}{2}, \frac{\pi}{3}\right)$ **8.** $\left(-2, \frac{\pi}{2}\right)$ **9.** $(4, \pi)$

In problems 10 to 13, graph the given polar point on polar graph paper, state another polar point that is equivalent to it, and give the exact rectangular coordinates of the point.

10. $\left(4, \frac{2\pi}{3}\right)$ **11.** $\left(2, \frac{3\pi}{2}\right)$ **12.** $\left(3, \frac{7\pi}{6}\right)$ **13.** $\left(2, -\frac{9\pi}{4}\right)$

In problems 14 to 17, Give polar coordinates for the rectangular point correct to the nearest hundredth. Express angles in degrees.

14. $(4, 2)$ **15.** $(-3, 5)$ **16.** $(-2, -5)$ **17.** $(4, -1)$

In problems 18 to 21, Find the exact distance between the given pair of points.

18. $(0, 0)$ and $(6, 45°)$

19. $(0, 0)$ and $(9, 75°)$

20. $(12, 68°)$ and $(5, 158°)$

21. $(5, 45°)$ and $(7, 105°)$

22. Find the three cube roots of $8i$. Express each in $a + bi$ form.

23. Suppose $z = (b + i)^2$, where b is a positive, real number. If $z = r \operatorname{cis} 60°$, find b.

24. Find the roots of $z^2 + 2z + 4 = 0$ and convert the answers to trigonometric form.

25. Euler's Formula (an equation most easily proved using series in calculus) states that $re^{i\theta} = r(\cos\theta + i\sin\theta)$.

a) Write i in the form $re^{i\theta}$.
b) Raise the result in part (b) to the ith power and simplify it to determine the exact value of i^i.
c) Show that $e^{i\pi} + 1 = 0$. (Notice anything interesting about this equation?)

26. Write a complex number that corresponds to an image rotation by 30° and a scaling by a factor of 2.

Section 5.3 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

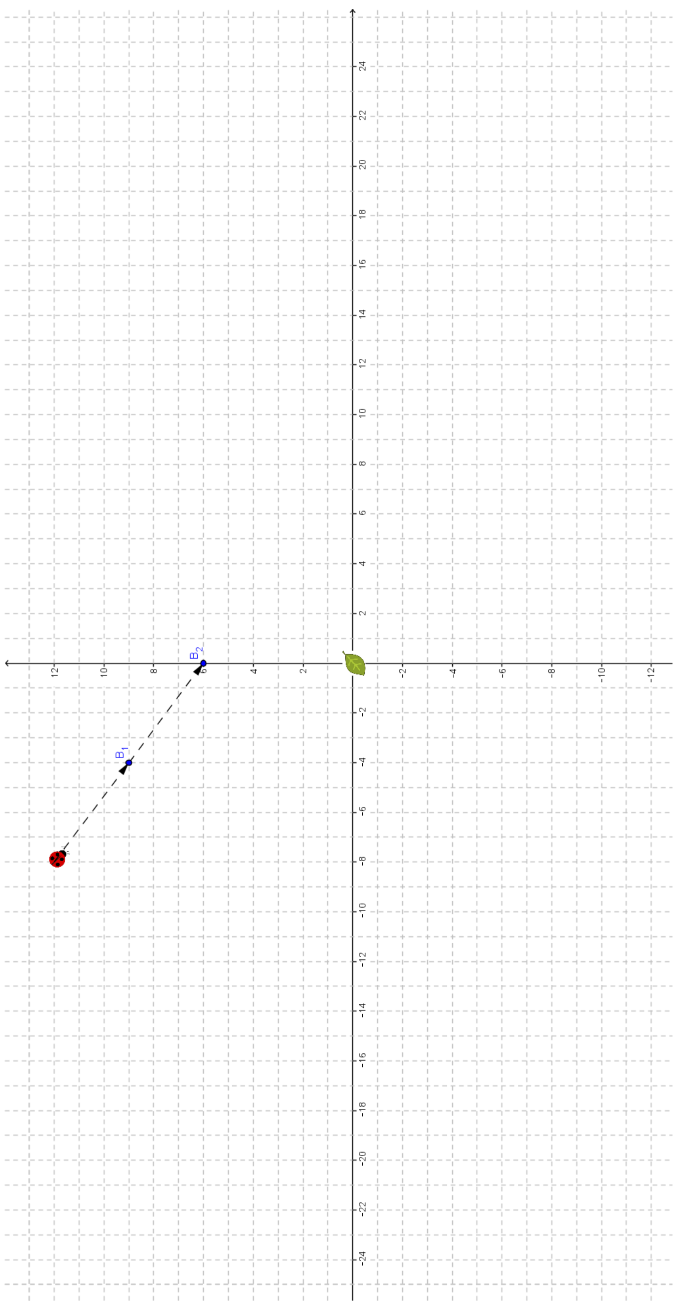
Questions for further exploration

6 Vectors

Section 6.1 Vector Notation, Terminology, and Equations of Lines

Exploration 6.1

A bug starts at the point $(-8, 12)$ and moves in a straight line as shown in the figure at a constant speed so that after one minute it is at point B_1, after two minutes it is at point B_2, and so on.



1. Find points B_3, B_4, B_5, and B_6.
2. How far does the bug travel each minute and, therefore, what is its speed?
3. Find an equation for the bug's x-coordinate as a function of time.
4. Find an equation for the bug's y-coordinate as a function of time.
5. Determine how far the bug is from the leaf at the origin when it is at each point B_1 through B_6.
6. Find an equation for the bug's distance from the leaf as a function of time.
7. Repeat items 1 to 6 for a bug that starts at the point $(3, 9)$ and is at the point $(-2, 7)$ after 1 second.
8. At what time does the bug in item 7 cross the x-axis?

•Parametric Equations• Taken together, the answers to item 3 and item 4 in Exploration 6.1 constitute the **parametric equations** for the line that the bug is crawling along. The name comes from the fact that the x- and y- coordinates for the points on the graph are each expressed as a function of the independent parameter t.

> **Think About It 6.1.1** Write a non-parametric equation for the graph produced by the parametric equations $x(t) = \cos t$ and $y(t) = \sin t$.

•Vector Notation and Terminology• Though the words speed and velocity are often used interchangeably in casual conversation, there is an important difference between them in math and physics. Speed is a **scalar** quantity, which means that it can be represented by a single number (along with its units), while velocity is a **vector** because it tells us both the direction of motion and the speed. Thus, it requires more than a single number to specify. A 2- or 3-dimensional **vector** can be visualized as an arrow in space which is fully defined by its **magnitude** (a.k.a. length) and its **direction**.

> **Think About It 6.1.2** In Exploration 6.1 you were asked to find the bug's speed. What was the bug's velocity?

Though it is possible to describe a vector's direction explicitly using bearings (such as $20°$ E of N), it is often more convenient to specify direction and magnitude together, albeit somewhat indirectly, by giving the **components** of the vector. The x-component of a vector is number which tells how many units you must move in the x-direction to get from the tail to the tip of the vector, and its sign tells whether you move in the positive or negative x-direction. The y-component gives us the same information about the y-direction, and if we work in three dimensions, it's easy to include a z-component as well. (In the discussion that follows we will sometimes use 2-dimensional vectors as examples, and sometimes use 3-dimensional vectors. The mathematics is identical either way, which is one of the things that makes vectors so useful.)

When we write the name of a vector, we put an arrow over it ($\vec{a}$) or, in print, we can put its name in bold (**a**). Absolute value bars around the name of the vector are used to represent the vector's magnitude: $|\vec{a}|$ (How is this related to the way we use absolute value bars around a scalar?)

There are three main notations used to specify a vector by giving its components. We'll get to the third of these notations later in this section, but the first two are **row form** and **column form**:

$$\vec{a} = \langle 1, -6 \rangle \qquad \text{or} \qquad \vec{a} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$

From either notation we can tell that $\vec{a}$ has an x-component of 1 and a y-component of -6. Note that once we have these components of $\vec{a}$, we can determine that

$$|\vec{a}| = \sqrt{37}$$

Think About It 6.1.3 Sketch a variety of vectors with various combinations of positive, negative, and 0 components. Give the component form and the magnitude of each one.

Think About It 6.1.4 It takes a bug 10 seconds to crawl from the point $(3, -5)$ to the point $(1, 2)$. Find the bug's speed (which should be a scalar) and velocity (which should be a vector).

While a vector is often given a single lowercase letter as its name, it is also common to specify vectors by identifying a "tail" point and a "tip" point (the end with the arrow). With this notation, a vector with its tail at point A and its tip at point B would be referred to as $\overrightarrow{AB}$. Knowing the components of a vector named this way does not tell you where either point is, but does allow you to find the location of one point if you know the other.

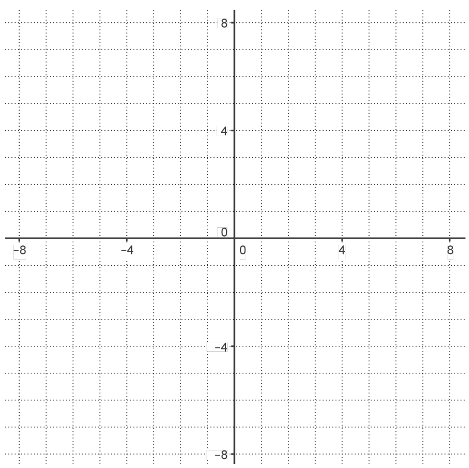
Think About It 6.1.5 If $\overrightarrow{PQ} = \langle -7, 2\rangle$ and the coordinates of Q are $(-3, -4)$, what are the coordinates of P?

•Vector Addition and Subtraction•

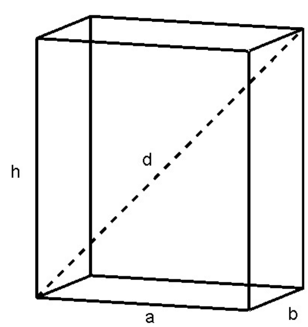
Exploration 6.2

Imagine that a bug on a coordinate plane starts at the point $S(3, -2)$ and walks to the point $D(-1, 5)$, where it stops for a drink. The bug then continues on to the point $N(7, 1)$, where it takes a nap.

Sketch a vector $\overrightarrow{SD}$ which represents the bug's displacement from its starting point to its drink locale and write $\overrightarrow{SD}$ in component form. Repeat for $\overrightarrow{DN}$, the bug's displacement from its drink locale to its nap locale, and for $\overrightarrow{SN}$, the bug's total displacement from its starting point to its nap. What does all this suggest for how we might define vector addition both geometrically and algebraically? How then would you define vector subtraction geometrically and algebraically?



Think About It 6.1.6 In the diagram below, find the length of diagonal d in terms of a, b, and h. (Show the work to convince any skeptics.) How is this related to the problem of finding the magnitude of a three dimensional vector? What does it suggest about how you would find the magnitude of a four dimensional vector?



•PositionVectors• When we talk about the **position vector** of an object, we mean the vector which tells us how to get from the origin of our coordinate system to the object. Typically the tail end of a position vector is denoted by O.

Think About It 6.1.7 If $\overrightarrow{OA} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, find $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

In TAI 6.1.7 you found that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OB}$. This is vector language for expressing the fact that we find the change in position of an object by subtracting the initial position from the final position. This works just as well in three dimensions (when the picture is much harder to draw) as it does in two dimensions.

Think About It 6.1.8 If $\overrightarrow{OP} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix}$, find $\overrightarrow{PQ}$ and $\overrightarrow{QP}$. How far does a bug flying directly from P to Q travel?

•Multiplication of a Vector by a Scalar• While we define addition (and subtraction) as an operation on two vectors, we define multiplication as an operation on a vector and a scalar:

$$n\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} na \\ nb \end{pmatrix}$$

Think About It 6.1.9 What is the effect on the magnitude and the direction of a vector when it is multiplied by a scalar?

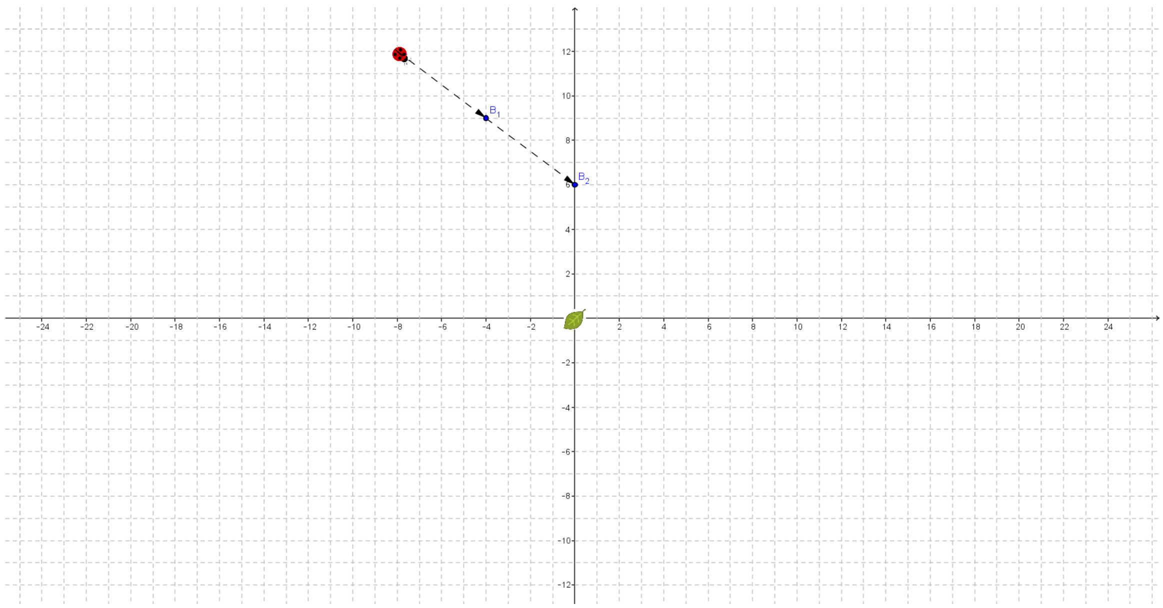
•Vector Equation of a Line• Remind yourself of the parametric equations for the line that the bug crawled along in Exploration 6.1. The same information when presented as a vector equation of the line looks like this:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 12 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{6.1}$$

or, equivalently,

$$\overrightarrow{OB} = \begin{pmatrix} -8 \\ 12 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{6.2}$$

Think About It 6.1.10 Choose a few different values of t and make a well-labeled diagram which shows how the two vectors on the right side of eq. (6.1) and eq. (6.2) can be added together to produce the vector on the left side.



Think About It 6.1.11 Write another vector equation for the line shown in TAI 6.1.10, but where both of the vectors on the right are different from those in eq. (6.1) and eq. (6.2).

Exploration 6.3

Calvin and Hobbes, who are standing on the ground, each throw a paper airplane in a straight line with the goal of smashing them into each other. Each airplane is released at time $t = 0$. (Assume that the planes have magical powers and each flies in a straight line forever, unless they bump into something. This is a comic strip, after all.) There is an $\times$ marked on the ground, which serves as the origin of the coordinate system in this problem, *i.e.*, its coordinates are $(0, 0, 0)$. The equations of the paths followed by the two planes, where distance is measured in feet, time is measured in seconds and the z-coordinate represents height, are as follows:

$$\text{Calvin's plane:} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

$$\text{Hobbes' plane:} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

E1. Find the position vector of each plane at $t = 0$, at $t = 1$, and at $t = 4$.

E2. If $\overrightarrow{CH}$ is the vector with its initial point at Calvin's plane and its terminal point at Hobbes' plane, find $\overrightarrow{CH}$ at $t = 0$, at $t = 1$, at $t = 4$, and in general.

E3. How high above the ground is Calvin's plane at the moment he releases it? After 1 second? In general?

E4. How high above the ground is Hobbes' plane at the moment he releases it? After 1 second? In general?

E5. Show that the distance from Calvin's plane to the $\times$ at the moment of release is $\sqrt{461}$ ft.

E6. Show that the distance from Hobbes' plane to the $\times$ at the moment of release is $\sqrt{110}$ ft.

E7. Show that the distance from Calvin's plane to Hobbes' plane at the moment of release is $\sqrt{465}$ ft.

E8. Tell how to find the distance between two objects if you know their 3D position vectors.

E9. Find the distance between the planes at $t = 1$, at $t = 2$, and at $t = 4$.

E10. Show that the speed of Calvin's plane is $\sqrt{21}$ ft/s.

E11. Find the speed of Hobbes' plane.

E12. Determine whether (A) the planes crash, (B) one plane passes directly over the other, or (C) neither of these things happens. Explain.

•Unit Vectors•

Definition 6.1.1 A **unit vector** is any vector with a magnitude of 1.

Think About It 6.1.12 Find the unit vector in the direction of $\vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Then use your result to help you determine the velocity vector of an object which is moving in the direction of $\vec{a}$ with a speed of 37 miles per hour.

The unit vectors that point in the same direction as the positive coordinate-system axes are given special names. Here we see these special names defined for the x, y and z directions:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(When working in just two dimensions, $\vec{k}$ becomes irrelevant, and we don't include the third row in $\vec{i}$ and $\vec{j}$.)

Think About It 6.1.13 Use vector addition and vector/scalar multiplication to write the vector $\vec{v} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ in column form. Do you see the third way of expressing a vector in component form?

Problem Set 6.1

1. If $\vec{v}$ has an initial point of $(5, 7)$ and a terminal point of $(-3, 1)$, find its magnitude and write it in component form.

2. Consider the vectors $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$.

a) On graph paper (though you don't actually need axes—why?), carefully draw $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AB}+\overrightarrow{BC}$, and $\overrightarrow{AB}+\overrightarrow{CB}$

b) What is the y-component of $\overrightarrow{AB}$

c) What is the magnitude of $\overrightarrow{AB}$

d) Find $|\overrightarrow{BC}|$

e) If point B had coordinates $(-1, 3)$, what would be the coordinates of point A and point C?

f) If point A had coordinates $(3, 10)$, what would be the coordinates of point C?

3. Repeat problem 2 if $\overrightarrow{AB} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$.

4. Repeat problem 2 if $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 3 \\ -13 \end{pmatrix}$.

5. Consider the points $A(2, -3, 4)$ and $B(-2, -2, 1)$. Find

a) $\overrightarrow{AB}$

b) $\overrightarrow{BA}$

c) $\left|\overrightarrow{BA}\right|$

d) the distance from A to B

e) the unit vector in the direction of $\overrightarrow{AB}$.

6. Repeat problem 5 if A has coordinates $(1, -2, 7)$ and B has coordinates $(-4, 2, 5)$.

7. Repeat problem 5 if A has coordinates $(-4, 0, 2)$ and B has coordinates $(2, -2, -3)$.

8. Consider the vectors $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j}$. Show how to find $\mathbf{u} = \mathbf{v} + 2\mathbf{w}$ both algebraically and geometrically.

9. At noon, a driver is 10 miles east and 4 miles south of her home and is driving along a straight road with a velocity (in miles per hour) which is given by the vector $\langle 24, 36\rangle$, where the positive x-direction is east and the positive y-direction is north.

a) Write a vector equation which gives the car's position vector relative to home (in miles) as a function of the time past noon.

b) Where is the car in relation to home at 12:15 pm?

c) Assuming your equation is valid for times earlier than noon as well, Where was the car in relation to home at 11:50 am?

d) If the speed limit is 40 mph, is the driver speeding?

e) Assuming your equation is valid for eternity, when is the driver closest to her home (and, for those willing to fire up a calculator, GeoGebra or Desmos, how far from home is she at that time)?

In problems 10 to 12, find a vector equation and a parametric equation for the line described.

10. the line through $(6, -5, 2)$ and $(2, 1, 4)$

11. the line through $(0, -3, 10)$ and parallel to the line where $x = 1 - 3t$, $y = -2 + 4t$, and $z = 3 - t$

12. the line through $(-8, 1, 2)$ and $(0, -1, 4)$

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13. If the point $(a, b, 5)$ is also on the line described in Problem 12, find the values of a and b.

14. Graph the function given by the parametric equations $x = 2t - 1$ and $y = -3t + 5$ if $1 \leq t \leq 4$. Then write an equation for y as a function of x.

15. Find a unit vector in the same direction as $\vec{v} = 40\vec{i} - 9\vec{j}$.

16. Find a unit vector in the same direction as $\vec{v} = 12\vec{i} - 5\vec{j}$.

17. If $\vec{r} = \langle \frac{1}{3}, -\frac{1}{2}, a \rangle$ is a unit vector, find a.

18. Find the velocity vector of a car moving at 60 miles per hour in the direction of $3\vec{i} - 4\vec{j}$

19. Find the velocity vector of an airplane whose speed is 210 mi/hr if the plane is flying in the direction $\langle 2, -2, 1 \rangle$.

20. Find the length of the longest diagonal of a box whose dimensions are 5 in $\times$6 in $\times$7 in.

21. If $\overrightarrow{CD} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$ and $\overrightarrow{CE} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$, find

a) $\overrightarrow{DE}$
b) $|\overrightarrow{CD} - \overrightarrow{CE}|$
c) a unit vector in the same direction as $\overrightarrow{CD}$
d) a vector of magnitude 10 in the opposite direction of $\overrightarrow{CD}$

22. Consider points $P(-1, 10)$ and $Q(5, 3)$ and vector $\overrightarrow{AP} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

a) Find $\overrightarrow{PQ}$, $\overrightarrow{QP}$, $\overrightarrow{AQ}$ and $\left|\overrightarrow{QP}\right|$
b) Find $\overrightarrow{PQ} + \overrightarrow{PA}$
c) Find the position vector for point A.
d) Find a vector equation for the line through A in the direction of $\overrightarrow{QP}$.
e) Find the parametric equations for the line through A in the direction of $\overrightarrow{QP}$.
f) Find the point-slope form of the equation for the line through A in the direction of $\overrightarrow{QP}$.
g) Find $\vec{v}$ if $\vec{v}$ is in the same direction as $\overrightarrow{AP}$ and has a magnitude of 20.
h) Find a unit vector in the direction of $\overrightarrow{AP}$
i) A bug starts at point A and crawls in the direction of $\overrightarrow{AP}$ at a speed of 2 units per second. Find
   1. the bug's velocity vector
   2. a vector equation for the bug's path as a function of the number of seconds since it was at point A
   3. the bug's distance from the origin 10 seconds after leaving point A
j) Find a vector perpendicular to $\overrightarrow{AP}$. Find another. Find yet another.
k) Try multiplying together the x-components of a pair of vectors that are perpendicular to each other. Now multiply together the y-components. What generalization can you make about x_1, y_1, x_2, and y_2 if $\langle x_1, y_1 \rangle$ is perpendicular to $\langle x_2, y_2 \rangle$?

23. Repeat problem 22 if P has coordinates $(-3, 7)$, Q has coordinates $(-2, -4)$, and $\overrightarrow{AP} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$.

24. Repeat problem 22 if P has coordinates $(4, -2)$, Q has coordinates $(-6, -7)$, and $\overrightarrow{AP} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$.

25. Find the value of p if $\begin{pmatrix} p \\ 4 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 3 \\ 1-2p \end{pmatrix}$.

26. If distance is measured in meters and time in seconds, then the position of a particular toy airplane t seconds after passing through point A is given by $\overrightarrow{p} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

a) the coordinates of point A
b) the speed of the plane
c) the position vector of the airplane half a second after passing through point A.
d) how long after passing through point A it will be until the airplane lands.

27. The position vector from a lighthouse to a boat is $\overrightarrow{r} = \begin{pmatrix} -10 \\ 30 \end{pmatrix} + t \begin{pmatrix} 6 \\ -8 \end{pmatrix}$, where t is the time in hours since 2:00 p.m. and distance is measured in km. Assume that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement of 1 km due east from the lighthouse, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement of 1 km due north from the lighthouse.

a) Give the parametric equations of the boat's path.
b) What is the boat's speed?
c) How far from the lighthouse is the boat at 2:00 p.m.?
d) What was the position vector from the lighthouse to the boat at noon?
e) At what time is or was the boat due east of the lighthouse?
f) The position vector from the lighthouse to a swimmer at 2:00 p.m. is $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and the swimmer's velocity is $\overrightarrow{i} - \overrightarrow{j}$. Write down a vector equation of the swimmer's path as a function of the number of hours after 2:00 pm.
g) Write the vector equation gives the displacement from the swimmer to the boat as a function of t.
h) At 2:00 p.m., the boat passes over a rock which is 0.2 km below the surface of the water and a seagull is 0.1 km directly above the swimmer. Write down the vector which gives the displacement from the seagull to the rock at 2:00 p.m.

28. Suppose a fly travels on a line in three dimensions. For a given time t, its x-coordinate, in feet and relative to a corner of a room, is given by the equation $x = 3 + t/2$ and its y-coordinate, also in feet, is given by $y = 5 - t/3$. If the fly passes through the points $A(6, 3, 2)$ and $B(7.5, 2, 6)$, determine an equation for its z-coordinate.

29. Repeat problem 28 if $x = 5 + t/4$, $y = 3 - t/3$, A has coordinates $(6.5, 1, 17)$, and B has coordinates $(8, -1, 35)$. Why might these coordinates of A and B not make sense in this particular context?

30. A spider moves on the xy-plane according to the vector equation $S = \langle -2, 5 \rangle + t\langle 1, -2 \rangle$. A fly moves on the same plane according to the vector equation $F = \langle 1, 1 \rangle + t\langle -1, 1 \rangle$. Where do their paths cross? (Hint: their paths don't reach the intersection point at the same time. Use two values, t_1 and t_2 to represent the parameter in their respective equations and solve by setting up a system of equations.)

31. Repeat problem 30 if $S = \langle -3, 1\rangle + t\langle 2, 3\rangle$ and $F = \langle 1, 3\rangle + t\langle 2, 7\rangle$.

32. Prove that using scalar multiples of the vectors $\mathbf{u} = \langle 1, 3\rangle$ and $\mathbf{v} = \langle 2, 5\rangle$ and vector addition, it is possible to reach any point on the xy-coordinate plane from the origin.

33. Prove that using the vectors $\mathbf{u} = \langle 1, 3\rangle$ and $\mathbf{v} = \langle 2, 6\rangle$, it is *not* possible to reach every point on the xy-coordinate plane from the origin.

34. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?

35. Is the line through $(-2, 5, -7)$ and $(2, -1, 1)$ parallel to the line through $(4, -11, -3)$ and $(-2, -2, -15)$?

In problems 36 to 38, Determine whether or not the lines intersect. If they do intersect, find their point of intersection.

36. L_1: $x = -6t_1$, $y = 1 + 9t_1$, $z = -3t_1$ and L_2: $x = 1 + 2t_2$, $y = 4 - 3t_2$, $z = t_2$.

37. L_1: $Q = \langle 2, 3, -1\rangle t_1 + \langle 1, 0, 2\rangle$ and L_2: $Q = \langle 1, 1, 3\rangle t_2 + \langle -1, 4, 1\rangle$

38. L_1: $Q = \langle 1, -1, 2\rangle t_1 + \langle 1, 1, 0\rangle$ and L_2: $Q = \langle -1, 1, 0\rangle t_2 + \langle 2, 0, 2\rangle$

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39. Find an equation of the line passing through $(-1, 5, 3)$ and parallel to $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ -2 \end{pmatrix} + t\begin{pmatrix} -11 \\ 3 \\ 3 \end{pmatrix}$.

40. Find an equation of the line passing through $(-4, 6, -1)$ and parallel to the line passing through both $(-2, -2, 5)$ and $(3, 0, 1)$.

41. Consider the vectors $\overrightarrow{u} = \langle -1, 5, -8\rangle$ and $\overrightarrow{v} = \langle -3, -3, 0\rangle$.

a) Find a non-zero vector $\overrightarrow{a}$ that is perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

b) Find the unit vector in the same direction as $\overrightarrow{a}$.

c) Find the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.

42. Consider $A(2, 3, 1)$, $B(4, -5, 21)$, and $O(0, 0, 0)$. Let $\overrightarrow{a} = \overrightarrow{OA}$ and $\overrightarrow{b} = \overrightarrow{OB}$.

a) Let $\overrightarrow{p} = \overrightarrow{AB}$. State $\overrightarrow{p}$.
b) Let S be the point $(3, -1, 11)$. The line L_1 goes through S and is parallel to $\overrightarrow{a}$. Give an equation for L_1.
c) The line L_2 has the equation $Q = \langle 5, 10, 10\rangle + t\langle -2, 5, -3\rangle$. Find where lines L_1 and L_2 intersect.

43. In three dimenstions, tell whether the statement is true or false.

a) Two lines parallel to a third line are parallel.
b) Two lines perpendicular to a third line are parallel.
c) Two planes parallel to a third plane are parallel.
d) Two planes perpendicular to a third plane are parallel.
e) Two lines parallel to a plane are parallel.
f) Two lines perpendicular to a plane are parallel.
g) Two lines are either parallel or they intersect.
h) Two lines perpendicular to a third line can be perpendicular to each other.

Section 6.2 The Angle Between Two Vectors and The Dot Product

•The Dot Product• The dot product is an operation defined in vector arithmetic which is different from any operation in real number arithmetic. While the notation may suggest to you that you should multiply the individual components, be sure to note that this is **not** how the operation is defined. Here is how it **is** defined:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Think About It 6.2.1 Is the result of the dot product of two vectors a vector or a scalar?

Think About It 6.2.2 Write down any 2-dimensional vector. Write down a vector which is perpendicular to your vector. Then find the dot product of these two vectors. Repeat for another pair of perpendicular vectors.

Exploration 6.4

The dot product is analagous to the product of two real numbers in that it represents a particular rectangular area associated with two vectors. Experiment with this geometric interpretation of the dot product at tube.geogebra.org/m/119554 and see if you can show that

$$\overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}|\,|\overrightarrow{v}|\cos\theta \qquad (\bigstar)$$

The starred equation, combined with the definition of the dot product, provides a convenient way for determining the angle between any two vectors.

Problem Set 6.2

1. If $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, and $\mathbf{w} = -2\mathbf{i} + \mathbf{j}$, find

a) $\mathbf{u} \cdot \mathbf{v}$
b) $\mathbf{v} \cdot \mathbf{w}$
c) the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest tenth of a degree

2. Repeat problem 1 if $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} - 4\mathbf{k}$, and $\mathbf{w} = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

3. Let $\overrightarrow{u} = \langle 2, 3, 5\rangle$ and $\overrightarrow{v} = \langle 6, -1, 2\rangle$.

a) State a non-zero vector that is perpendicular to both $\langle 2, 3, 5\rangle$ and $\langle 6, -1, 2\rangle$ simultaneously.
b) State the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.

4. Repeat problem 3 if $\overrightarrow{u} = \langle 3, -1, 2\rangle$, and $\overrightarrow{v} = \langle 2, 5, -3\rangle$.

5. Find an equation for a plane that is perpendicular to $\langle 2, 1, 5\rangle$.

6. Prove that for any vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$, it is true that $\overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w} = \overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w})$.

7. Use vectors to prove that the diagonals of any rhombus are perpendicular.

8. A tram car has been knocked off a (horizontal) bridge by Doc Ock, but Spiderman saves the day by shooting two strands, S_1 and S_2, from the tram car to the bridge to stop its fall. Spiderman is on the tram at point T, 120 feet below point P on the bridge. The two strands are connected to the bridge 140 feet apart at points A and B. $AP = 90$ feet and $BP = 50$ feet. If the weight of the tram car is 10000 pounds, find the tension on each strand.

Section 6.3 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

Problem Set 6.3

1. If $\overrightarrow{a} = \langle -2, 3, -6 \rangle$ and $\overrightarrow{b} = \overrightarrow{i} - 5\overrightarrow{j} + 2\overrightarrow{k}$, find

a) $\overrightarrow{b} - 10\overrightarrow{a}$
b) $|\overrightarrow{a}|$
c) a unit vector in the same direction as $\overrightarrow{a}$
d) a vector of magnitude 5 in the direction opposite $\overrightarrow{a}$

2. Write a vector equation and a set of parametric equations for the line $y - 8 = -2(x + 11)$.

3. A robot is moving in a straight line along along a coordinate plane where each unit is one foot. Its position t minutes after being turned on is given by the vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \end{pmatrix} + t\begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

a) Determine the robot's distance from the origin when it was turned on and state its speed and velocity.
b) How long after being turned on did the robot cross the x-axis and what was its position vector then?

4. Given $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}$, and $\overrightarrow{CB} = \begin{pmatrix} 2 \\ -9 \\ -4 \end{pmatrix}$, find $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\left|\overrightarrow{CB} - \overrightarrow{AO}\right|$.

7 Counting and Probability

Section 7.1 Counting

If you're not a card player, necessary background information for problems involving a "standard deck" is that such a deck is made up of 52 cards. Each card has one of four suits: Clubs, Diamonds, Hearts, or Spades. The Clubs and Spades are black, while Hearts and Diamonds are red. Within each suit there are 13 cards: one card with each number from 2 through 10, plus an Ace, a King, a Queen, and a Jack.

Problem Set 7.1

In problems 1 to 8, evaluate the expression without the assistance of technology.

1. ${}_{2018}C_0$

2. ${}_{2018}C_{2018}$

3. ${}_{2018}C_1$

4. ${}_{2018}C_{2017}$

5. $\dfrac{101!}{99!}$

6. $\dfrac{12!}{9!}$

7. $\dfrac{10!}{7!\,3!}$

8. $\dfrac{(n+1)!}{n!} - n$

9. The standard U.S. coin denominations are 1, 5, 10, 25, 50 cents, and 1 dollar. Assume a machine randomly picks one coin from each of two bags that contain one of each type of coin.

a) How many ways are there to pick two pennies or two nickels?
b) How many ways are there to pick two coins such that their total value is at least 70 cents?
c) What is the probability that the two coins will have the same denomination? (Hint: probability is the number of ways to get what you want to happen divided by the number of ways anything could happen.)

10. The standard U.S. coin denominations are 1, 5, 10, 25, 50 cents, and 1 dollar. Assume a machine randomly picks two coins from a single bag that contains two of each type of coin.

a) How many ways are there to pick two pennies or two nickels from the bag?
b) How many ways are there to pick two coins such that their total value is at least 70 cents?
c) What is the probability that the two coins will have the same denomination? (Hint: probability is the number of ways to get what you want to happen divided by the number of ways anything could happen.)

11. What makes problem 9 and problem 10 different from each other? What assumptions did you make in answering the questions in those problems?

12. What's the number of ways to pick a president, vice-president, and secretary from a group of 10 people?

13. What's the number of ways to pick a committee of 3 people from a group of 11 people?

14. How many four digit numbers have at least one 7?

15. Write a counting problem where the solution would be $\dfrac{30!}{20!\,10!}$.

16. How many ways can we arrange the letters in the word INTUITION?

17. How many ways are there to arrange 12 children in a circle for a game of ring-around-the-rosy?

18. You want to order pizzas for a club activity. The local pizza chain you order from offers 5 different types of pizzas. If you need to order 15 pizzas, how many different combinations of pizzas could you order?

19. In the game of 3-Toed Pete, three cards are dealt from a standard deck.

a) How many distinct 3-card hands are possible?
b) How many 3-card hands contain at least one Jack?
c) How many 3-card hands contain one Ace, one King, AND one Queen?
d) How many 3-card hands contain only spades?
e) Write a 3-Toed Pete problem where the solution would be $\dfrac{26!}{3!\,23!}$.

20. 82 people were surveyed and asked if they showered in the morning and then asked if they showered in the evening. 56 people said they showered in the morning and 43 people said they showered in the evening. 21 people said yes to both questions. How many of these 82 people do not shower in either the morning or the evening?

21. How many one, two, or three scoop ice cream cones are possible if there are 15 possible flavors, flavors aren't allowed to repeat, and the order of the flavors in each cone does not matter?

22. A state's license plate system is 3 letters followed by 3 numbers. The letters cannot have repeats and the first letter cannot be a vowel. The 3 numbers cannot all be zero simultaneously. What's the number of total possible license plates?

23. In the game of Texas Hold'em, two cards are dealt from a standard deck.

a) How many 2-card hands are possible?
b) How many 2-card hands contain cards of different colors?

24. Given positive integers n and r where $r \leq n$, what is the largest possible value for $\dfrac{{}_nC_r}{{}_nP_r}$? Use the formulas to prove your answer true.

25. How many integers between 1 and 1000, inclusive, are divisible by 3 or 7 but not by 4?

26. Let sequence a be defined by the formula $a_n = \dbinom{2n}{n}$. List the first 5 terms of a_n (n starts at 1).

27. Expand $(2x - \sqrt{2})^6$.

Section 7.2 A Plethora of Potentially Perplexing Probability Problems

Problem Set 7.2

In problems 1 to 6, determine your answers to the questions and then discuss your answers with others in your class.

1. A hat contains three quarters. One is normal, one has two heads, the other has two tails. One quarter falls out of the hat onto a table, heads up. What is the probability the other side is also a head?

2. A bag contains a bean that is known either to be white or black (with equal probability). A white bean is added to the bag, the bag is shaken, and one bean is taken out at random. It is white. What is the chance that the next bean taken out of the bag will be white?

3. A bag contains two beans, each of which are known either to be white or black (with equal probability). One bean is taken out at random. It is black. What is the probability that both of the beans that were initially in the bag were black?

4. You're on a game show and there's three doors to pick from. Behind one of these doors is a car, behind another is a goat, and behind the remaining door there's a small pile of uninteresting pennies. You want the car for some reason, not the goat. You pick door 1, and the host reveals that there's a goat behind door 2. Do you switch to door 3? Explain.

5. You are given two sealed envelopes. In each is a sum of money. You are told that one envelope has exactly twice as much money in it as the other. You are allowed to open one and count the money. Then, before the other envelope is opened, you make a decision: you may keep the money or you may take the money in the other envelope. What is the best strategy: keep the money or switch envelopes?

6. Suppose you're playing a game in which a fair coin is tossed until heads comes up. If heads appear on the first toss you win \$1. If heads first appear on the second toss you win \$2. If heads first appear on the third toss you win \$4, and so on. What would be a fair amount for you to pay in order to have the privilege of playing this game?

In problems 7 to 9, translate the sequence of symbols into English.

7. $P(A) + P(A') = 1$

8. $P(A \cap B) = P(A)P(A|B)$

9. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

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10. When are two events mutually exclusive? Explain in words as well as by using an equation.

11. When are two events independent? Explain in words as well as by using an equation.

12. A box contains 3 red stones and 5 green stones. 2 stones are randomly picked, one after the other, without replacement.

a) Are the events "second stone is red" and "first stone is green" independent? Prove your answer.
b) Are the events "first stone is red" and "second stone is green" mutually exclusive? Prove your answer.
c) What is the probability that 1 red stone and 1 green stone are selected?

13. You have 4 dice: one 4-sided, one 6-sided, one 8-sided, and one 10-sided, each numbered from 1 to the number of sides of the die, each side being equally likely on each die. The four dice are placed into a bag. One of the dice is then randomly selected and rolled. What is the probability that the result of the roll is a 6?

14. Given 3 dice, how many ways can they be rolled so that exactly two of them have the same value?

In problems 15 to 18, two cards are drawn without replacement from a standard deck of 52 cards.

15. Draw a tree diagram for this situation.

16. Determine the probability that both cards are black.

17. Determine the probability that the two cards are different colors.

18. Determine the probability that the two cards are either both aces or both kings.

Problems 19 to 21 refer to this situation: A drawer contains 4 red, 6 blue, and 8 yellow socks. Two socks are drawn randomly and without replacement.

19. Draw a tree diagram for the situation.

20. What is the probability that two socks of the same color are selected?

21. What is the probability that a red or a blue sock is picked first, given that a yellow sock was picked second?

Problems 22 to 24 refer to this situation: Box A contains 6 red balls and 2 green balls. Box B contains 4 red balls and 3 green balls. A fair cubic die with faces numbered 1 through 6 is thrown. If a 2 or a 3 is rolled, a ball from Box A is selected. Otherwise, a ball from Box B is selected.

22. Draw a tree diagram for the situation.

23. Calculate the probability that the ball selected is red.

24. Given that the ball selected is red, calculate the probability it came from box B.

Problems 25 to 27 refer to this situation: An ambidextrous person has a fair coin in one hand and a normal, 6-sided die in the other.

25. Are the events "coin comes up heads" and "die comes up 4 or 5" mutually exclusive? Explain your answer.

26. Are the events "coin comes up tails" and "die comes up 4 or 5" independent? Explain your answer.

27. What is the probability that the coin comes up heads **and** the die comes up 2 or 3?

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28. 8 students roll three normal dice (sides numbered 1 through 6) with the hopes of winning candy. Their goal is to roll exactly 2 sixes on the three dice.

a) If you are one of the students, what's the probability that you will roll exactly two sixes on your three dice?

b) What's the probability that exactly 3 of the 8 students will roll exactly two sixes?

29. Suppose a student attempts to roll exactly three 6's on four normal six-sided dice. What's the probability of the student being successful?

30. Now suppose a class full of 14 students attempts the same feat attempted by the student in Problem 29. What is the probability that exactly two students are successful? At most two students?

31. Three emeralds and five sapphires are placed in a jar. One gem is randomly chosen, its type is noted, and is then set aside. Another gem is drawn from the jar, and its type is also noted.

a) Make a tree diagram of the given situation.
b) Write the expression "the probability that the second stone is an emerald, if the first gem is a sapphire" in symbols according to your tree diagram and the format mentioned above. Then use your tree diagram to determine the probability in question.
c) Determine the probability that the second gem is an emerald.
d) Are the events "the first gem is an emerald" and "the second gem is a sapphire" independent? Explain.
e) Determine the probability that the first gem was a sapphire, given that the second gem was an emerald.
f) Determine the probability that the first gem was an emerald, given that the second gem was a sapphire.

32. A box contains 13 red marbles, 14 clear marbles, and 5 green marbles. Two marbles are drawn from the box without replacement.

a) What is the probability that the first marble is a green one, given that the second marble is a red one?
b) What is the probability of either a red or a green marble being picked second?
c) What is the probability of a clear marble being picked second, if a red or a green marble was picked first?
d) What is the probability that a clear and a green marble is selected?
e) What is the probability that a clear or a green marble is selected, given that a red marble was picked second?

33. We can use the idea of conditional probability to assess the strength of various cause-and-effect relationships. For example, doctors use a list of symptoms given by their patient in order to narrow down the cause of the patient's illness. Their years of study have enabled them to quickly analyze conditional probability: what is the probability that the patient has disease D (the cause of their illness) given that they have symptoms X and Y (the effects of their illness)?

During a flu epidemic, 40% of a certain population have the flu. Of those with the flu, 85% of them have high temperatures. Of course, it is also possible for someone to have a high temperature and not have the flu, though the probability of this happening is 17% for this population.

a) Draw a tree diagram for this situation. To make things easier, a bit of notation: given an event A, the symbol A' represents event A not happening. That is, $P(A) = 1 - P(A')$.
b) What percentage of the population has a high temperature?
c) What is the probability that a person in this population has a flu, given that they have a high temperature?

34. We can use probability to determine the likelihood of a certain ailment in a patient given his/her symptoms. Suppose the probability of mononucleosis in a particular population is 0.27. Suppose furthermore that *if* a person of that population has mono, the probability that they have a fever is 0.63, but if a person does *not* have mono, the probability that they have a fever is 0.11.

a) Draw a tree diagram for this situation.
b) What's the probability that someone of that population has a fever but doesn't have mono?

c) What's the probability that someone of that population has a fever?
d) What's the probability that someone has mono given that they have a fever? Discuss how strong of an indicator a fever is of someone having mono based on your answer.

35. If a normal, 6 sided die comes up 1 or 6, you lose \$3. If it doesn't, you win your result in dollars. How much money would this game cost to play for this game to be considered fair?

36. If a normal, 6 sided die comes up 1 or 6, you win \$3. If it doesn't, you win \$$x$. If the game costs \$1 to play, what is the value of x that would make this game "fair" according to expected value?

37. Economists use a measurement of utility similar to something called 'happiness units', or HU. That is, they assign a certain number of HU for a given event–the happier an event makes a person, the higher the HU payoff. If the HU payoff is negative, it corresponds to a person becoming unhappy. (This assignment allows them to avoid the varying ways sums of money could be considered useful to people in differing financial situations.)

Artemov is deciding whether or not to perform an illegal, though tempting action, and decides to use expected value along with the notion of HU described above. A chart depicting a list of events and their estimated HU payoff and probability of occurring is shown. What is the expected value, in terms of HU, for Artemov? **Interpret** your result.

Event	Probability of Event	Payoff for Event
Not getting caught	.75	200 HU
Getting caught and receiving probation	.12	-150 HU
Getting caught and losing job	.05	-500 HU
Getting caught and going to prison	.08	-1500 HU

38. A six-sided, normal die and a normal coin is used in a carnival game. If the die roll is a 1 or a 2, the coin is tossed once and the player wins \$4 if the result of the coin toss is heads. If the die roll is 3 or higher, the coin is tossed *twice* and the player wins \$4 if *both* tosses come up heads. The game costs \$2 to play. Does the player have an advantage in this game?

39. The game C-Lo is played with three normal dice rolled simultaneously, and the best roll to get is a 4-5-6 (in any order). What is the probability that someone will get the best roll on their first try?

40. A jar contains 14 green balls and 8 blue balls. Seven balls are picked from the jar without replacement. What is the probability that four of them are blue and three of them are green?

41. The letters E, N, O, S, T are randomly arranged. What is the probability that the word "STONE" will be created?

42. In a history class with 25 students, there are 18 students who take robotics and 13 students who take epistemology. If every student takes at least one of these two classes, what is the probability that a student is taking epistemology given that they are taking robotics?

43. A student intends to roll a normal, 6-sided die 18 times and write down the results in the order they occur. Which sequence of results below is more likely to happen, and why?

123456123456123456 632243236323464634

44. A student has finished rolling a normal, 6-sided die 18 times and has written down the results in the order they occurred. Which sequence of numbers is more likely to be the sequence that has been written down, and why?

123456123456123456 632243236323464634

45. A bag contains 4 red and n green balls. If two balls are chosen without replacement, the probability of two red balls is 2/15. Find the value of n.

46. A penny, a nickel, a dime, a quarter, a half-dollar, and a dollar coin are in a bag. If two coins are randomly chosen, what's the probability their value sums to over 30 cents?

Section 7.3 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

8 Proof

Section 8.1 Arriving at Results

"I have had my results for a long time, but I do not yet know how to arrive at them."

– Karl Friedrich Gauss

Problem Set 8.1

1. Prove that for any positive integer n, one of n, $n+2$, or $n+4$ must be divisible by 3.
2. Prove that the formula for the sum of the first n positive integers is $S_n = \dfrac{n(n+1)}{2}$.
3. Prove that the sum of any four-digit number and the number obtained by reversing the digits is divisible by 11.
4. Prove that for any vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$, it is true that $\overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w} = \overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w})$.
5. Prove that using the vectors $\overrightarrow{u} = \langle 1, 3 \rangle$ and $\overrightarrow{v} = \langle 2, 5 \rangle$, it is possible to reach any point on the xy-coordinate plane from the origin.
6. Prove that if k is odd, the sum of *any* k consecutive integers (not just the first ones) will always be divisble by k. Hint: Use the claim in problem 2.
7. Prove that all prime numbers larger than 3 are either of the form $6k+1$ or $6k-1$ for some integer k.
8. Using the previous claim, prove that the square of any prime number larger than 3 is of the form $24n+1$ for some integer n.
9. Prove that $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for all positive integers n. Hint: break down the fractions.
10. Prove that in a set of any 5 distinct integers, there will be a 3-element subset whose members sum to a multiple of 3.
11. Prove that there are infinitely many primes.
12. Prove that there are more real numbers between 0 and 1 than there are positive integers.

Section 8.2 Proof by Induction

"If we have no idea why a statement is true, we can still prove it by induction."

– Gian-Carlo Rota

Problem Set 8.2

Try proving the claims in problems 1 to 22 by inducting on n. In general, these claims are true for all positive integers unless otherwise specified, but every generalization has at least one exception...

1. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.
2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\displaystyle\sum_{k=0}^{n} (k \cdot k!) = (n+1)! - 1$.
4. The sum of the first n positive odd numbers is n^2.
5. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.
6. $(1 + \frac{1}{1})(1 + \frac{1}{2})(1 + \frac{1}{3}) \ldots (1 + \frac{1}{n}) = n + 1$.
7. $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \ldots (1 - \frac{1}{n}) = \frac{1}{n}$.
8. $3^n > 2^n$.
9. $7^{2n} - 48n - 1$ is divisible by 2304.
10. $11^n - 6$ is divisible by 5.
11. $8^n - 5^n$ is divisible by 3.
12. 5 is a factor of $2^{2n-1} + 3^{2n-1}$.
13. $2^n > n^2$ for $n \geq 5$.
14. Let $a_1 = 5$, $a_2 = 13$, and $a_n = 5a_{n-1} - 6a_{n-2}$. Then, $a_n = 2^n + 3^n$.
15. Let $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Then F_{4n} is divisible by 3.
16. Let $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Then $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$.
17. $1 + x + x^2 + x^3 + \cdots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$.
18. The sum of the internal angles in an n-gon, where $n \geq 3$, add up to $180(n-2)$.
19. Let $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Then $F_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$, where ϕ is the golden ratio, $\dfrac{1 + \sqrt{5}}{2}$.

20. $(1+x)^n \geq 1+nx$, if $x > -1$.

21. $\binom{2n}{n} \geq 2^{n+1}$ for all positive integers $n \geq 3$.

22. All cats are black.

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23. Let $f_0(x) = \dfrac{x}{x+1}$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$. Find an explicit formula for $f_n(x)$ and prove the formula true by induction.

24. Find the smallest k such that $k! > 5^k$. Then, prove that $n! > 5^n$ for all integers $n \geq k$ by induction.

Section 8.3 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

9 A Little Bit of Calculus

Section 9.1 Limits

Problem Set 9.1

In problems 1 to 16, find the limits and, when possible, provide an algebraic justification.

1. $\lim\limits_{x\to\infty} \dfrac{x-\sqrt{x^3-3}}{3x-5}$

2. $\lim\limits_{x\to\infty} \dfrac{x^2-\sqrt{x^3-3}}{3x^2-5}$

3. $\lim\limits_{x\to-\infty} \dfrac{x^2-\sqrt{x^4-3}}{3x-5}$

4. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

5. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos x}{\cot x}$

6. $\lim\limits_{\theta\to\frac{\pi}{3}} \dfrac{\sin\theta}{\csc\theta}$

7. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}}{(x-9)^4}$

8. $\lim\limits_{x\to 1} \dfrac{x^6-1}{x^8-1}$

9. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

10. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

11. $\lim\limits_{\triangle x\to 0^+} \dfrac{(x+\triangle x)^2+(x+\triangle x)-(x^2+x)}{\triangle x}$

12. $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{x}$

13. $\lim\limits_{x\to 0} \dfrac{\sin(\pi x)}{6x}+2$

14. $\lim\limits_{x\to 0^-} x^3\cos\left(\dfrac{2}{x}\right)$

15. $\lim\limits_{x\to\infty} (\dfrac{1}{x-1}-\dfrac{1}{x})$

16. $\lim\limits_{x\to w} \dfrac{w^{\frac{1}{2}}-x^{\frac{1}{2}}}{w-x}$

---

17. Evaluate $\lim\limits_{x\to 4} \dfrac{|x-4|}{x-4}$. Explain your answer and include a graph.

18. Find a value a such that $\lim\limits_{x\to -2} \dfrac{3x^2+ax+a+3}{x^2+x-2}$ exists, and evaluate the limit.

19. If $\lim\limits_{x\to a}[f(x)+g(x)]=2$ and $\lim\limits_{x\to a}[f(x)-g(x)]=1$, find $\lim\limits_{x\to a} f(x)g(x)$.

20. Find integers a and b such that $\lim\limits_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x}=1$. (Hint: rationalize the numerator and use some of the properties of limits to solve.)

Section 9.2 Derivatives

Problem Set 9.2

1. Explain why if $y = f(x)$ has a horizontal asymptote as x approaches infinity, then $\lim_{x\to\infty} f'(x) = 0$.
2. Find the equation for the line tangent to the curve $y = 3x^2 - 4x + 3$ where $x = 1$.
3. Find the parabola $y = ax^2 + bx$ such that the equation for the tangent line to the curve through $(1, 1)$ is $y = 3x - 2$.
4. If $f(x) = 3x^3 - 2x + 6$, sketch the graph of $y = (f(x))^{-1}$, $y = f^{-1}(x)$, and $y = f'(x)$.
5. Graph $y = \dfrac{4x^2 - 1}{x^2 + 1}$ and state what x-values correspond to the local minima and maxima.

Section 9.3 Chapter Review

Problems I should try again

Key terms and concepts

Reminders to self

Questions for further exploration

10 Extra Practice

Section 10.1 Review Problems

1. Write two different equations in *point-slope* form for the line of slope 4 which passes through the point $(10, -29)$.

In problems 2 to 21, evaluate the expression without a calculator.

2. $\left(\frac{4}{9}\right)^{-2}$

3. $\left(-\frac{1}{100,000}\right)^{3/5}$

4. $16^{-3/2}$

5. $(-27)^{2/3}$

6. $27^{-2/3}$

7. $\frac{3^0}{8^{-5/3}}$

8. $\frac{2 \cdot 7^0}{36^{-1/2}}$

9. $\log_3\left(\frac{1}{81}\right)$

10. $\log_6 0$

11. $\log_7(-49)$

12. $\log_8(2)$

13. $\log_{0.5}\left(\frac{1}{8}\right)$

14. $\log_8 32$

15. $\log_4 32$

16. $\log_2 \sqrt[5]{16}$

17. $\log_{1000} 100$

18. $\log_{0.2} 25$

19. $\log \sqrt[3]{\frac{1}{100}}$

20. $\sum_{n=3}^{101} i^n$

21. $\sum_{n=6}^{99} i^n$

---

22. Solve without a calculator: $\log_2 2 + \log_2 x + \log_2(x-1) = 2\log_2(x+3)$

23. Find the exact solution(s) to $2^{3x+1} = 5^{x+4}$.

24. Find the x- and y-intercepts of the function $f(x) = 2^x - 20$. If either does not exist, explain how you know. For any value which is not an integer, give an exact value and list the two consecutive integers that the value is between.

25. Find the x- and y-intercepts of the function $f(x) = 3^x - 90$. If either does not exist, explain how you know. For any value which is not an integer, give an exact value and list the two consecutive integers that the value is between.

26. Find the x- and y-intercepts of the function $y = 10^x - 1200$. If either does not exist, explain how you know. For any value which is not an integer, give an exact value and list the two consecutive integers that the value is between.

In problems 27 to 28, simplify the expression so that the result is a single fraction which doesn't itself contain fractions.

27. $\dfrac{\frac{x}{2x+3} - \frac{1}{3}}{\frac{3}{x} - 1}$

28. $\dfrac{\frac{1-2y}{x-y} - \frac{1}{x}}{2xy - y}$

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29. State the formula for finding the sum of the first n terms of a geometric sequence and show how to use the formula to find the sum of the first 6 terms of the geometric sequence $3, 6i, -12, \ldots$

30. Write an equation for the constant function $g(x)$ which passes through the point $(-2, 6)$ and state its domain and range using interval notation.

31. Write exactly two limit expressions to describe the end behavior of the function $f(x) = 4 + \left(\frac{2}{5}\right)^x$

32. The function $h(x)$ is graphed at right. Find the following, estimating where necessary.

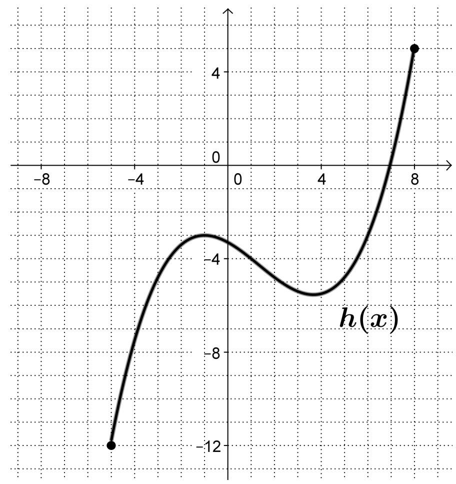
a) $(h \circ h)(1)$

b) the interval(s) on which h is decreasing

c) the solution(s) to the equation $h(x) = -\frac{2}{3}x + 1$

d) the solution(s) to the inequality $h(x) \geq -3$

e) the domain of $k(x)$ if $k(x) = \sqrt[4]{-h(x)}$



In problems 33 to 36, state the domain of the function.

33. $f(x) = \sqrt{4-x}$

34. $g(x) = \dfrac{x-1}{x^2+5x+6}$

35. $f(t) = \dfrac{\sqrt{t-2}}{t^2-6t+5}$

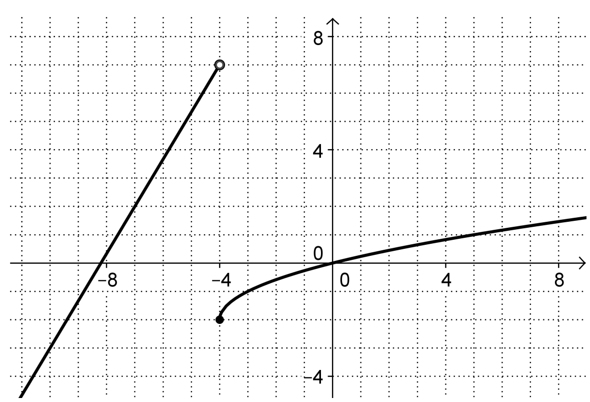
36. $g(t) = \dfrac{t+1}{t^{2/3}-9}$

37. Solve without the aid of technology: $\dfrac{(x-4)^2(x-3)}{(5-x)(x+7)} \geq 0.$

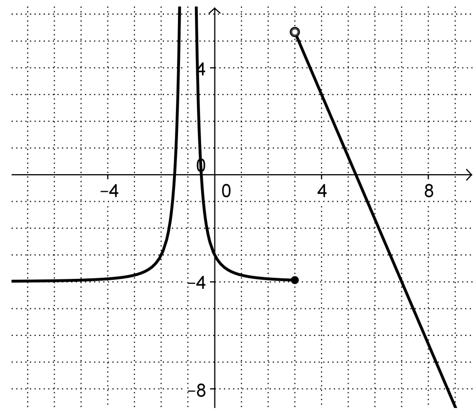
38. Consider the functions $Q(x) = -\sqrt{x+4}$ and $R(x) = \dfrac{2}{x^2-9}$

a) State the domains of Q and R.
b) Find $(R \circ Q)(x)$ and simplify.
c) Write the limit expressions that describe the end behavior of Q and R.

39. The graph of the function $f(x)$ is shown at right. The domain of f is $(-\infty, \infty)$ and each of its pieces is a transformation of a power function. Write a piecewise equation for f.



40. The graph of the function $g(x)$ is shown at right. The domain of g is all reals except -1 and it can be thought of as a piecewise-defined function with two pieces, each of which is a transformation of a power function. Write an equation for g.



41. Consider the function $f(x) = \dfrac{1}{(x-6)^2}$.

a) Write down functions $P(x)$, $Q(x)$, and $R(x)$ such that$P\left(Q\left(R(x)\right)\right) = f(x)$. (None of your functions should be simply x.)
b) Write down limit expressions which describe the end behavior of f.
c) Give the equation(s) of any asymptotes of f. If there are none, explain how you know.

42. Sketch a rough graph of $g(x) = (x-3)^2(x+1)^3(x+4)$, ensuring that your graph includes all intercepts and reflects an understanding of end behavior and general shape. Describe the end behavior of g using limit notation.

43. Identify all key features (*e.g.*, holes, asymptotes, intercepts) of $f(x) = \dfrac{x^2-3x-4}{x^2+x}$ and use the information to carefully graph f.

44. Identify all key features (*e.g.*, holes, asymptotes, intercepts) of $g(x) = \dfrac{x^2 - -4}{x^2+3x+2}$ and use the information to carefully graph g.

45. Consider the function $g(x) = \dfrac{1}{x+2} - 3$.

a) Write down functions $P(x)$, $Q(x)$, and $R(x)$ such that$P\left(Q\left(R(x)\right)\right) = g(x)$. (None of your functions should be simply x.)
b) Write down limit expressions which describe the end behavior of g.
c) Give the equation(s) of any asymptotes of g. If there are none, explain how you know.

46. If $g(-5) = 8$ and $g(2) = -6$, find the coordinates of **two points** other than $(-5, 8)$ and $(2, -6)$ that would be on the graph of

a) $y = |g(x+1)| - 7$
b) $y = g(x)$ if g is odd
c) $y = \dfrac{5g(-x)}{2}$
d) $y = g(x)$ if g is even

47. If $g(1) = 12$ and $g(-4) = -10$, find the coordinates of **two points** other than $(1, 12)$ and $(-4, -10)$ that would be on the graph of

a) $y = g(x)$ if g is odd

b) $y = 1 - \dfrac{g(4x)}{2}$

c) $y = g(x)$ if g is even

d) $y = |g(-x)|$

48. Consider the function $P(x) = -3(x + 20)^2 + 12$.

a) Tell what type of local extremum P has, determine what this local maximum or minimum value is, and state the value of x where it occurs.

b) Find the average rate of change of P from $x = -20$ to $x = -17$.

c) If $Q(x) = \sqrt{P(x)}$, state the domain of $Q(x)$.

49. Consider the function $P(x) = 28 - 7(x - 18)^2$.

a) Tell what type of local extremum P has, determine what this local maximum or minimum value is, and state the value of x where it occurs.

b) Find the average rate of change of P from $x = 16$ to $x = 19$

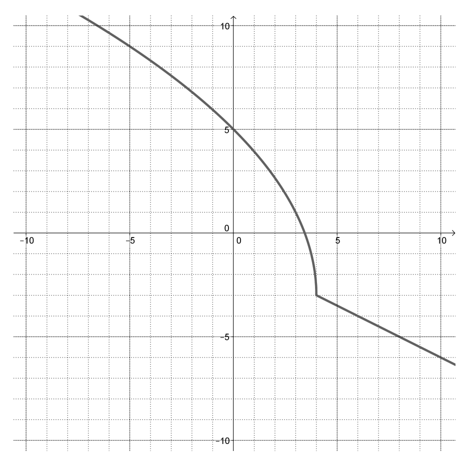
c) If $Q(x) = \dfrac{x}{\sqrt{-P(x)}}$, state the domain of $Q(x)$.

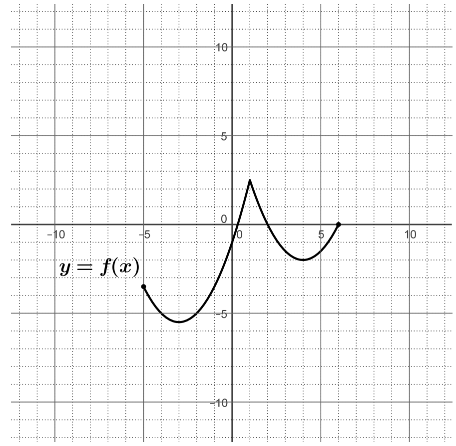
50. Consider the function $Q(x)$, whose graph is shown at right. Assume the domain of Q is $(-\infty, \infty)$.

a) Find the average rate of change of Q on $[-5, 4]$.

b) Find the average rate of change of Q on $[55, 1000]$.

c) Graph $y = Q^{-1}(x)$.





51. The graph of f is shown at right at right.

a) Graph $a(x) = f\left(-\frac{x}{2}\right)$

b) Graph $b(x) = -f(|x|)$

c) Graph $c(x) = -1 + 2f(x+3)$

d) On what intervals is f increasing? decreasing?

52. Evaluate the difference quotient $\dfrac{f(3+h) - f(3)}{h}$ if $f(x) = \dfrac{1}{x^2}$.

53. Write an equation for a function which is not invertible.

54. For what values of a will $v(t) = a^t$ be a decreasing function?

55. For what values of a will $v(t) = -a^t$ be a decreasing function?

56. Is the function $f(x) = x(2^x - 8)$ invertible? Explain how you know.

57. Consider the function $g(x) = 5 - 3^x$.

a) State the domain and range of g using interval notation, find its x-and y-intercepts, and give the equations of any asymptotes.
b) Find $g^{-1}(x)$.

58. Consider the function $f(x) = \log_2 x$.

a) Graph $y = f(x)$, plotting at least 4 points very accurately.
b) State the domain, range, and equations of all asymptotes of f.
c) Write an equation for $h(x)$ if $(h \circ f)(x) = x$ for all x in the domain of f.
d) If $g(x) = \dfrac{x-8}{\sqrt{-f(x)}}$, what is the domain of g?

59. Consider the function$f(x) = \log_{0.5} x$.

a) Graph $y = f(x)$, plotting at least 4 points very accurately.
b) State the domain, range, and equations of all asymptotes of f.
c) Write an equation for $h(x)$ if $(h \circ f)(x) = x$ for all x in the domain of f.
d) If $g(x) = x^2\sqrt{f(-x)}$, what is the domain of g?

60. Determine the APY, to the nearest thousandth of a percent, on an investment which earns 6.5% compounded

a) quarterly

b) continuously

61. Determine the APY, to the nearest thousandth of a percent, on an investment which earns 5.75% compounded

a) monthly

b) continuously

62. The half-life of fluorine-18 is 109.77 minutes. How long (to the nearest hundredth of a minute) will it take for a 200.00 mg sample to decay to 15.00 mg?

63. Over the course of the day on Monday, December 15, 2014, Ford's stock lost 4.74% of its value.

a) If the price at the stock at the end of Monday was \$4.28, what was its price (to the nearest cent) at the start of the day?

b) If the stock were to continue to decline at this rate every day of the week, what would its price (to the nearest cent) be at the end of the day on Friday, December 19?

64. Rewrite the equation $A(x) = 2000e^{-0.024x}$ in the form $A(x) = A_o\,(0.8)^{x/b}$. Then describe a situation that these equations might model, indicating precisely what $A(x)$ and x represent, and what the parameters 2000, -0.024, and b tell you about the situation you've imagined.

65. Write an equation for a quadratic function with a y-intercept of 2 and x-intercepts of -4 and -10 and state the range of the function.

66. Write an equation for the parabola that passes through the points $(5, 3)$, $(-9, 3)$ and $(1, -5)$ and state its y-intercept.

67. Consider the function $g(x) = 3 + 5x - 2x^2$

a) Write the equation for g in vertex form.

b) If $R(x) = \dfrac{-1}{\sqrt{g(x) - 3}}$, state the domain of R.

c) Find the average rate of change of g from $x = a$ to $x = a + h$. (Simplify your answer fully.)

68. Give the equations of the vertical and horizontal asymptotes and the coordinates of any holes in the graph of $R(x) = \dfrac{(x^2 - 4)(2x + 7)}{5(x + 2)(x^2 - 3)}$. If any of these do not exist, say so.

In problems 69 to 80, evaluate the expression without a calculator.

69. $\sin\left(-\dfrac{2\pi}{3}\right)$

70. $\cos\dfrac{7\pi}{6}$

71. $\sec\dfrac{5\pi}{6}$

72. $\cot 5\pi$

73. $\tan^2\dfrac{5\pi}{3}$

74. $\csc(13\pi)$

75. $\cot(330°)$

76. $\tan^{-1}\sqrt{3}$

77. $\tan^2\dfrac{3\pi}{3}$

78. $\arccos\left(-\dfrac{1}{2}\right)$

79. $\tan(\cos^{-1} 0.25)$

80. $\cos^2\left(\dfrac{3\pi}{8}\right) - \sin^2\left(\dfrac{3\pi}{8}\right)$

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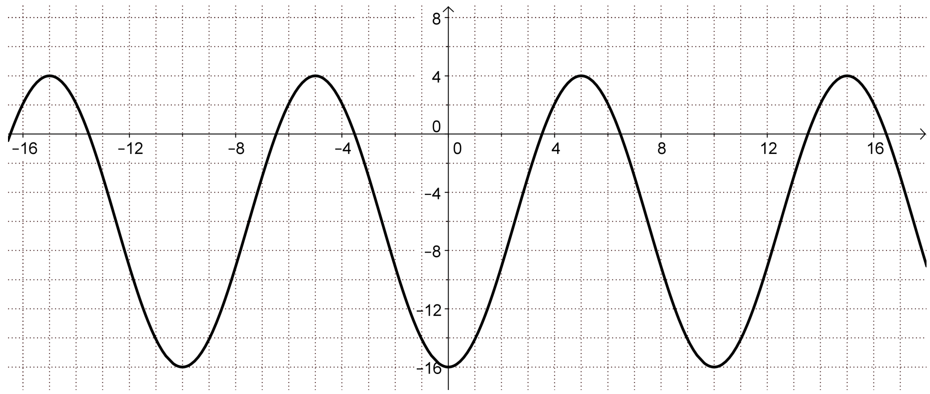
81. Make a careful graph of $y = 2\sin\left(3\pi\left(x - \dfrac{\pi}{6}\right)\right)$.

82. If $\tan\alpha = -\frac{3}{5}$ and the terminal side of α is in the fourth quadrant, find $\sin\alpha$ and $\cos\alpha$.

83. State the amplitude and period of the function shown and write an equation for it.



84. Which of the following functions is changing most rapidly at $t = 2$? Write a few sentences in defense of your choice. (Assume t is measured in seconds in each case.)

$f(t) = \frac{2}{3}(t-3) + 8$ $\qquad$ $g(t) = \cos \dfrac{\pi t}{2}$ $\qquad$ $h(t) = 2^t$

$x(t)$ where x represents the position of a bug moving along the x-axis at the rate of 1.5 units per second.

85. Carefully graph $f(x) = 2\cot x$ over the interval $[-2\pi, 2\pi]$ State the domain, range, and period of f.

86. Consider the function $f(x) = \sin(2x)$.

a) Graph $y = f(x)$ for $-2\pi \le x \le 2\pi$
b) Graph $y = f(|x|)$
c) Find functions g and h so that $(g \circ f \circ h)(x) = \csc x$.
d) Find the average rate of change of f on the interval $\left[\dfrac{\pi}{3}, \dfrac{13\pi}{3}\right]$.
e) Find the average rate of change of f on the interval $\left[\dfrac{\pi}{4}, \dfrac{11\pi}{4}\right]$.

87. Verify: $\cos(2x) + 2 - 2\cos^2 x = 1$

88. Verify: $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$

89. Solve without the aid of technology: $4\sin^2(3x) - 2 = 0$

90. Find all solutions to the equation $1 - 5\cos(16x) = 3$, giving approximations correct to three places after the decimal.

91. Make a careful graph of $h(\theta) = 10\cos^{-1}\left(\dfrac{\theta}{2}\right)$.

92. Show how to rewrite this expression as one of the six standard trig functions: $\tan\beta + \dfrac{\cos(-\beta)}{1-\sin(-\beta)}$

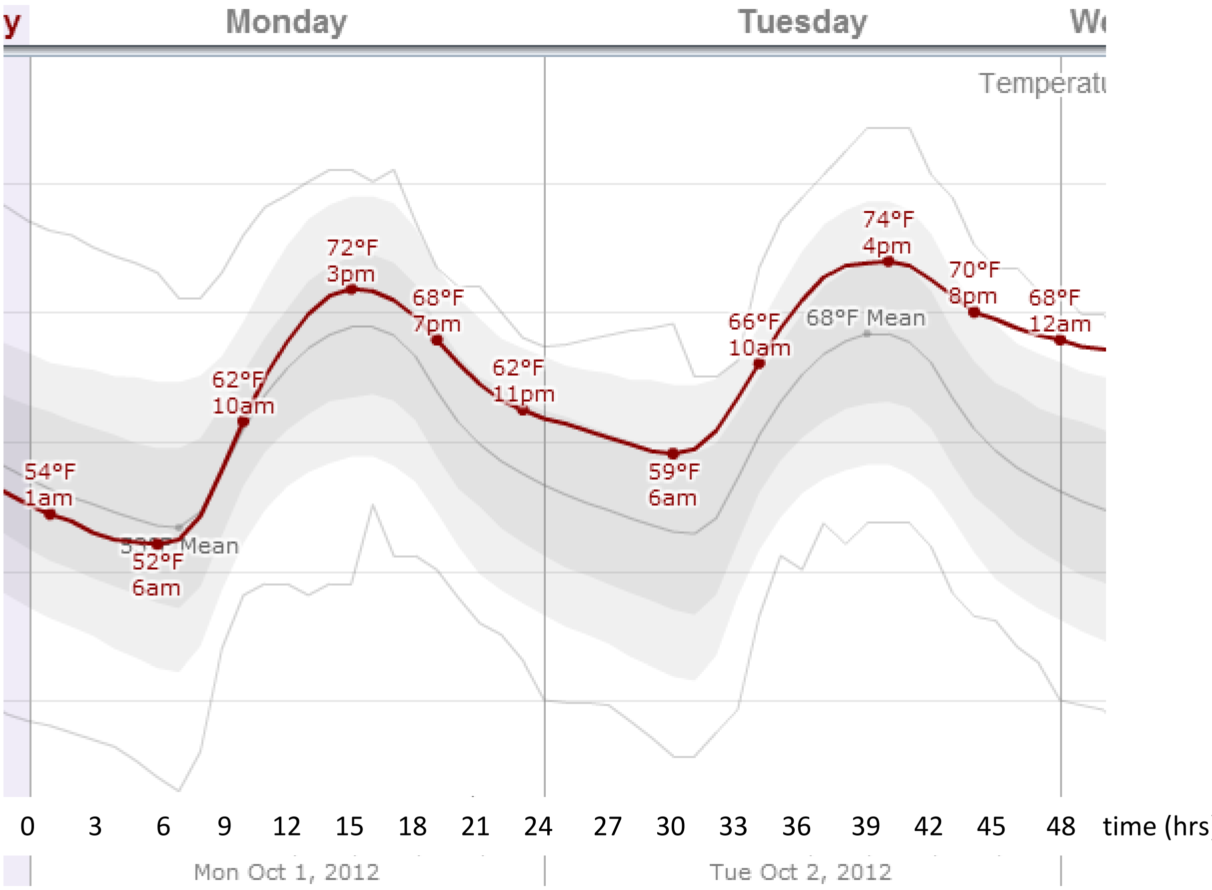
93. $(3\vec{i} - \vec{j}) \cdot (7\vec{i} + 2\vec{j})$

94. In this problem, let $\vec{i}$ represent a displacement of 1 km east and $\vec{j}$ represent a displacement of 1 km north. A car whose initial position vector is $-10\vec{i} + 5\vec{j}$ moves at 60 km/hr in the direction $3\vec{i} - 4\vec{j}$.

a) Find the car's velocity vector.
b) Write a vector equation for the car's position as a function of time.

A Miscellaneous Worksheets

A Temperature vs. Time Graph



Graph from WeatherSpark.com

This darkest line in this graph is the function $T(x)$, where T is the forecasted temperature in °F and x is the time in hours since the beginning of October 1, 2012.

1. Identify the interval(s) on which T is increasing.

2. Identify the interval(s) on which T is decreasing.

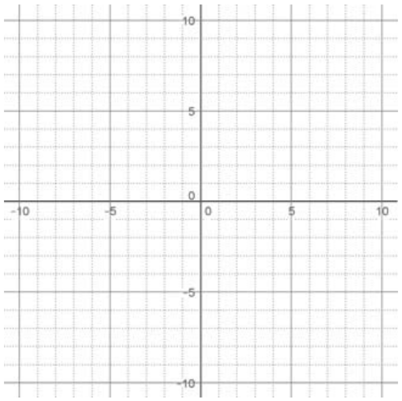
3. Identify the local maximum and minimum values of T and state the values of x where they occur.

4. Find the average rate of change of T from $x = 6$ to $x = 15$. Include units.

5. Find the average rate of change of T from $x = 10$ to $x = 30$. Include units.

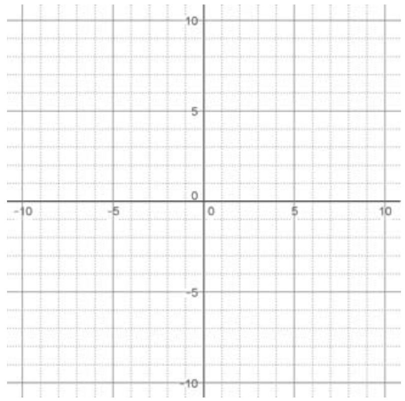
Absolutely Vital Functions 1: Power Functions

$f(x) = x^0$ (2 pts. required)



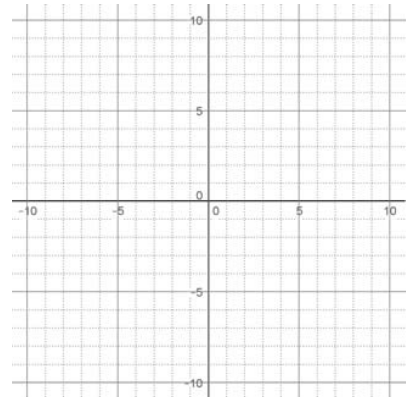
domain: range:

$f(x) = x$ (2 pts. required)



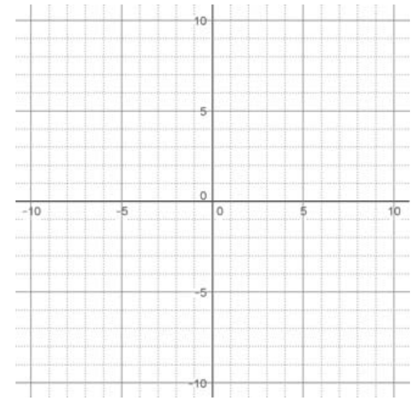
domain: range:

$f(x) = x^2$ (7 pts. required)



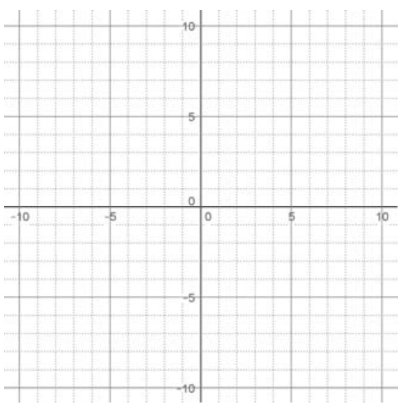
domain: range:

$f(x) = x^3$ (5 pts. required)



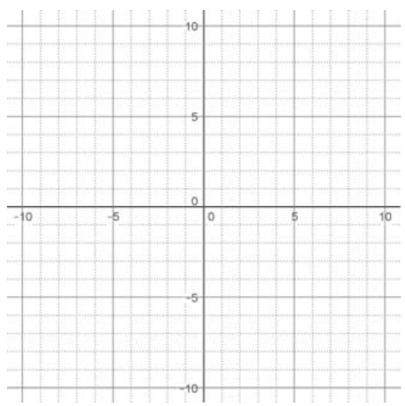
domain: range:

$f(x) = x^{1/2}$ (4 pts. required)



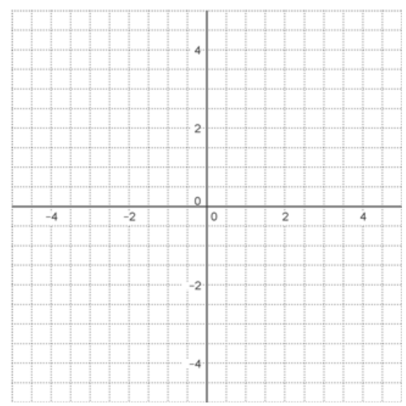
domain: range:

$f(x) = x^{1/3}$ (5 pts. required)



domain: range:

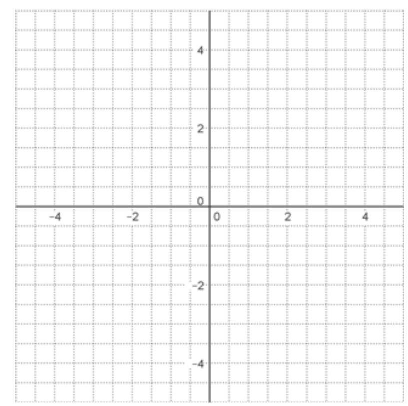
$f(x) = x^{-1}$ (6 pts. required)



domain: range:

equations of asymptotes:

$f(x) = x^{-2}$ (6 pts. required)

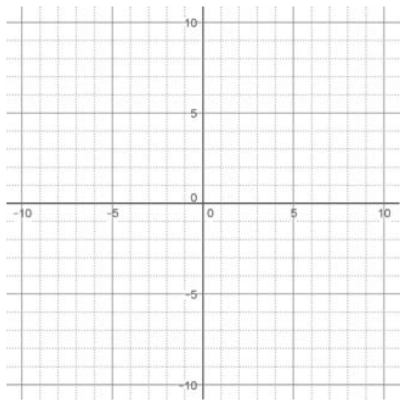


domain: range:

equations of asymptotes:

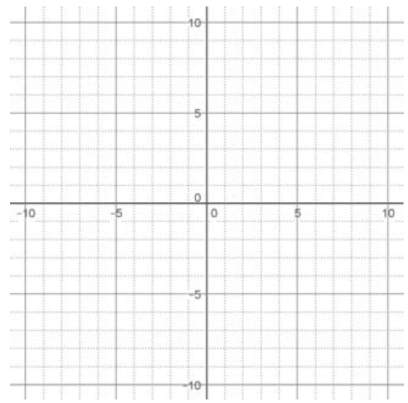
Absolutely Vital Functions 1: Power Functions

$f(x) = x^0$ (2 pts. required)



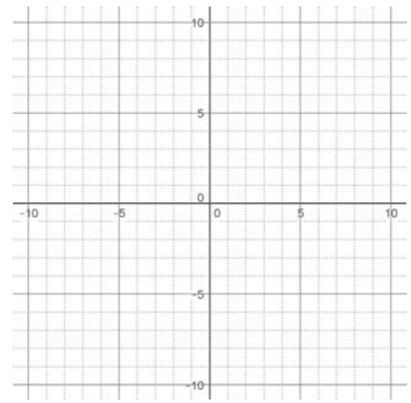
domain: range:

$f(x) = x$ (2 pts. required)



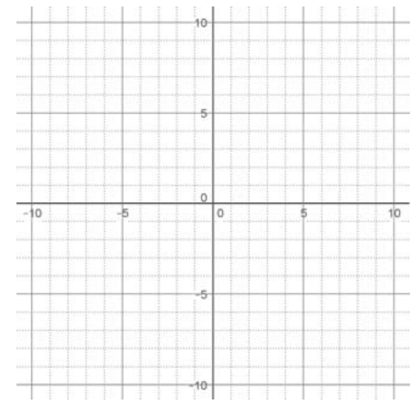
domain: range:

$f(x) = x^2$ (7 pts. required)



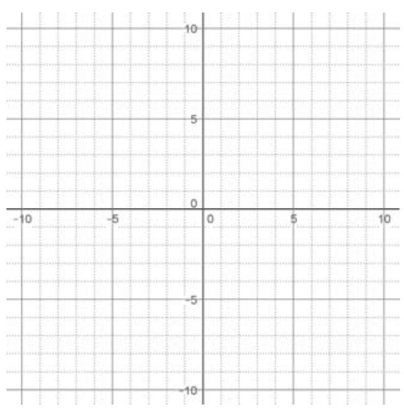
domain: range:

$f(x) = x^3$ (5 pts. required)



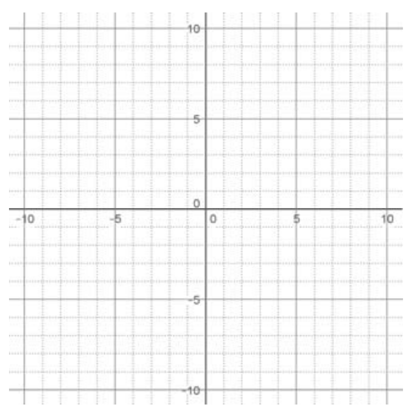
domain: range:

$f(x) = x^{1/2}$ (4 pts. required)



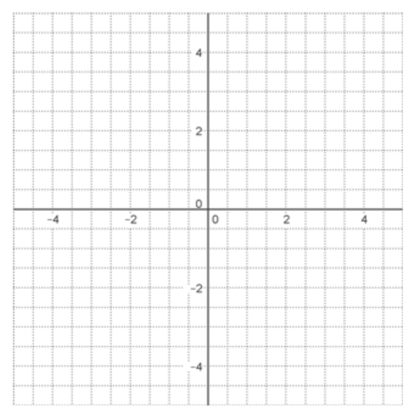
domain: range:

$f(x) = x^{1/3}$ (5 pts. required)



domain: range:

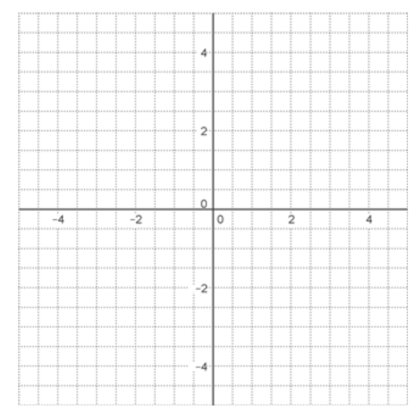
$f(x) = x^{-1}$ (6 pts. required)



domain: range:

equations of asymptotes:

$f(x) = x^{-2}$ (6 pts. required)

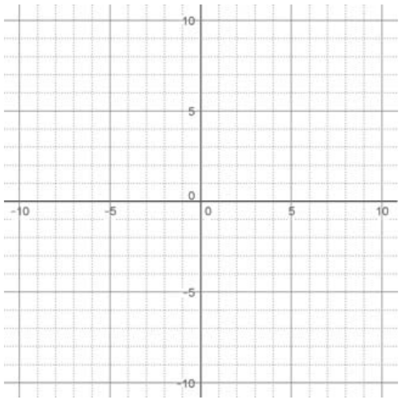


domain: range:

equations of asymptotes:

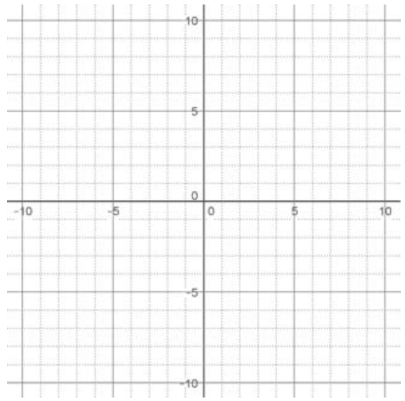
Absolutely Vital Functions 1: Power Functions

$f(x) = x^0$ (2 pts. required)



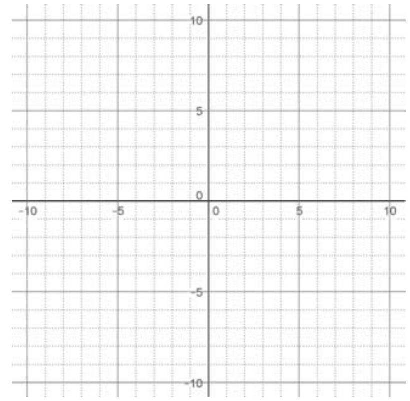
domain: range:

$f(x) = x$ (2 pts. required)



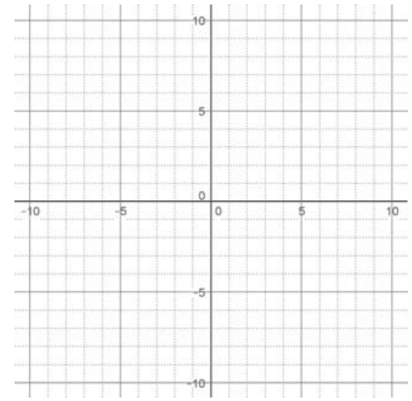
domain: range:

$f(x) = x^2$ (7 pts. required)



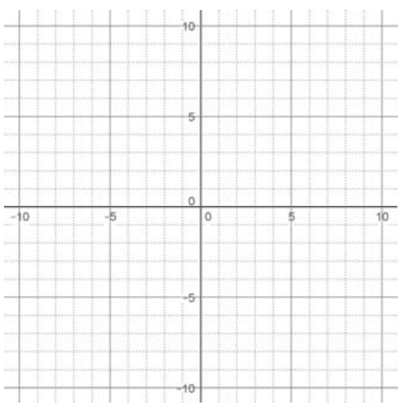
domain: range:

$f(x) = x^3$ (5 pts. required)



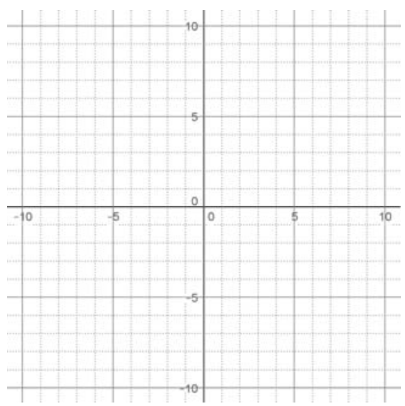
domain: range:

$f(x) = x^{1/2}$ (4 pts. required)



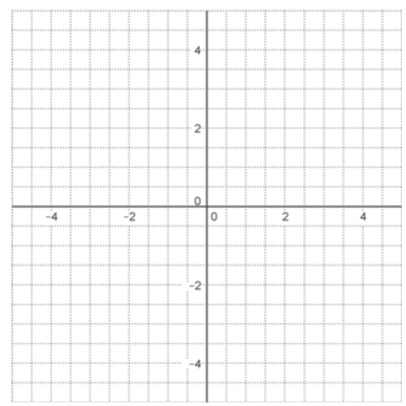
domain: range:

$f(x) = x^{1/3}$ (5 pts. required)



domain: range:

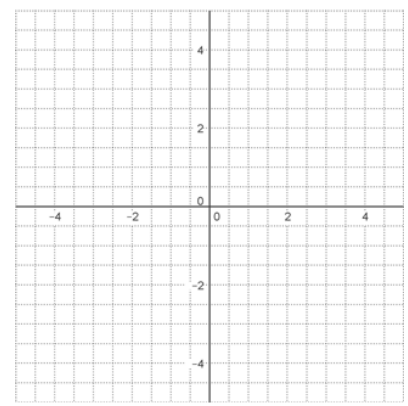
$f(x) = x^{-1}$ (6 pts. required)



domain: range:

equations of asymptotes:

$f(x) = x^{-2}$ (6 pts. required)

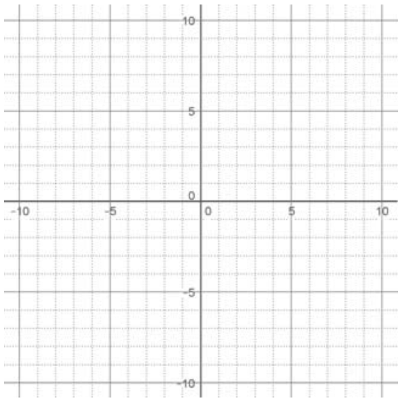


domain: range:

equations of asymptotes:

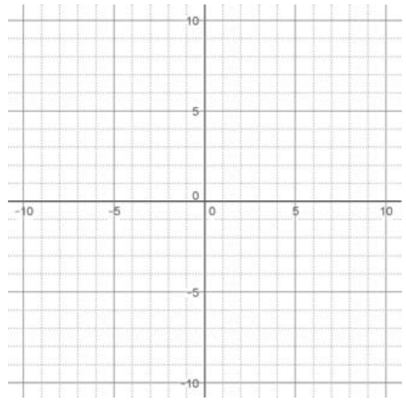
Absolutely Vital Functions 1: Power Functions

$f(x) = x^0$ (2 pts. required)



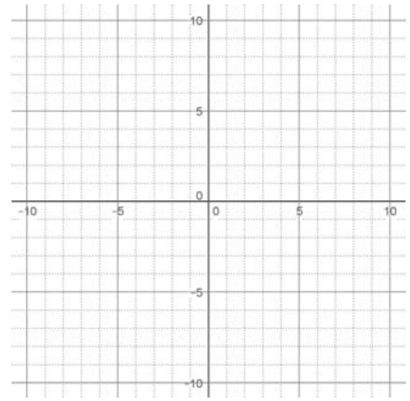
domain: range:

$f(x) = x$ (2 pts. required)



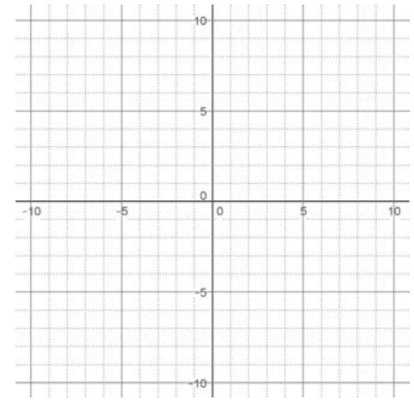
domain: range:

$f(x) = x^2$ (7 pts. required)



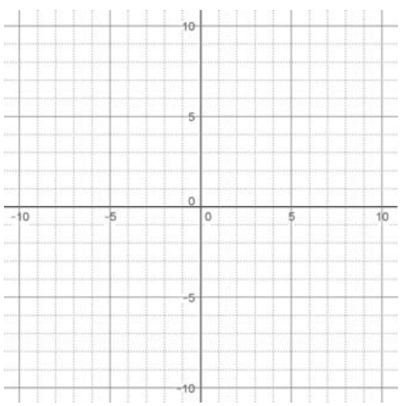
domain: range:

$f(x) = x^3$ (5 pts. required)



domain: range:

$f(x) = x^{1/2}$ (4 pts. required)



domain: range:

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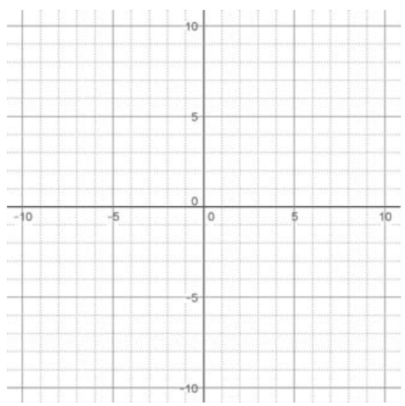
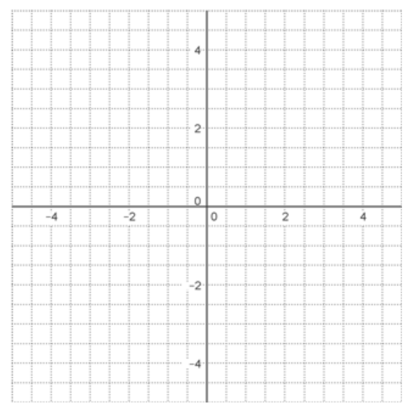


domain: range:

$f(x) = x^{-1}$ (6 pts. required)



domain: range:

equations of asymptotes:

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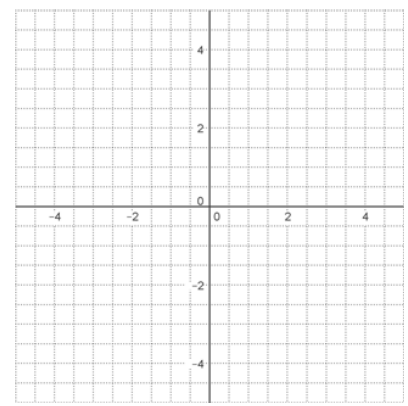


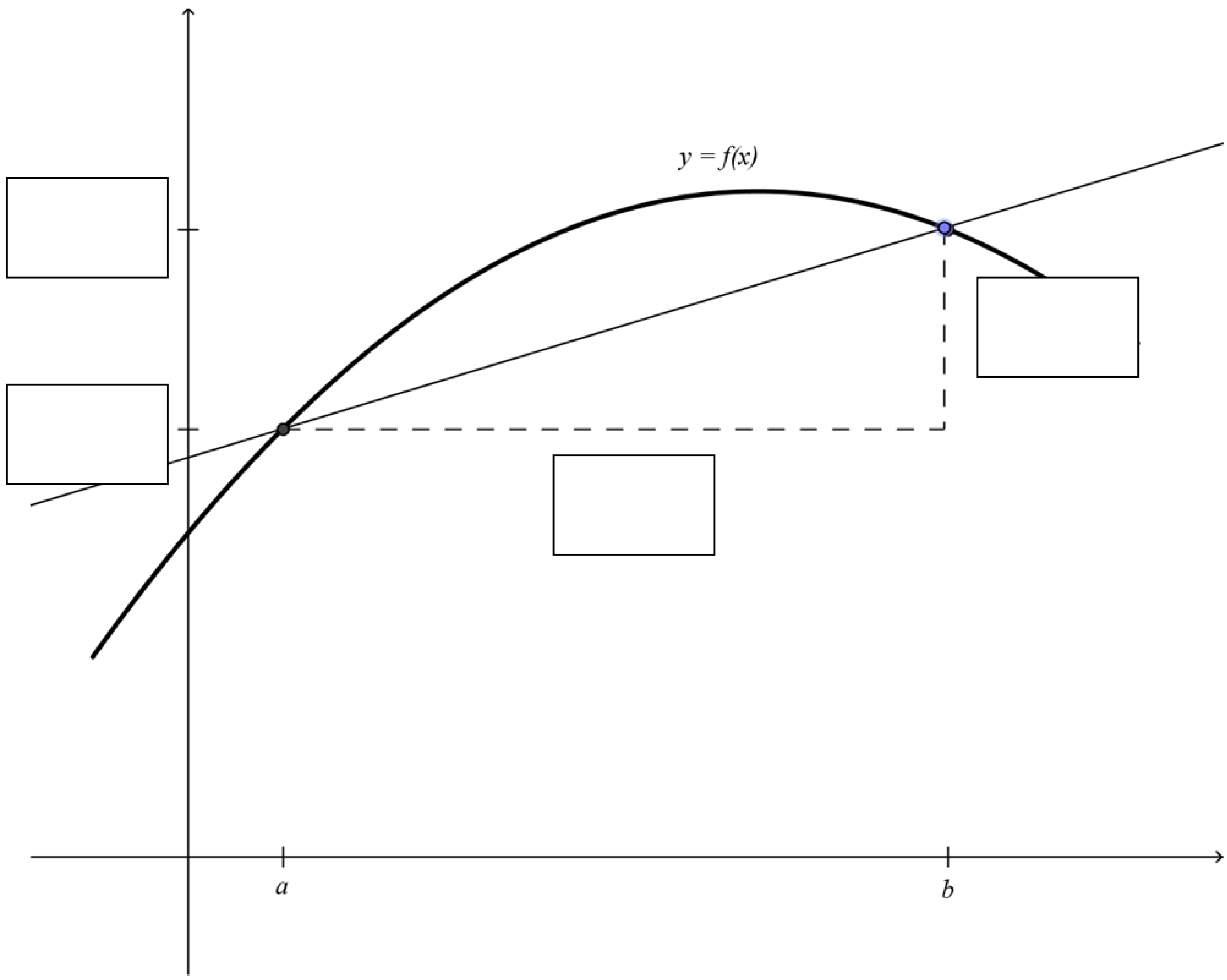
domain: range:

equations of asymptotes:

Average rate of change, version 1

Place each label in the appropriate box on the diagram. Then write an expression which gives the average rate of change of the function.

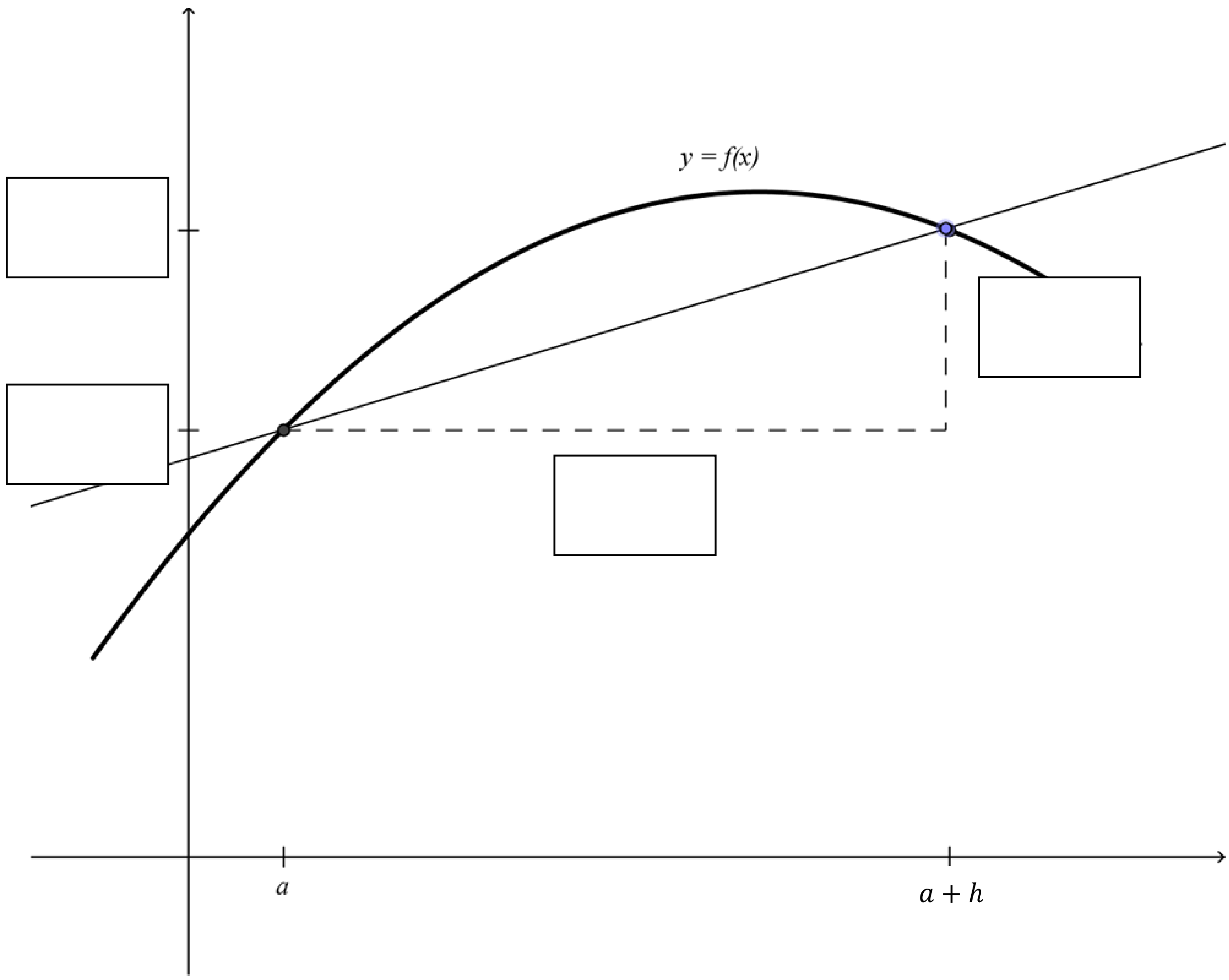
$b - a$ $\qquad$ $f(b)$ $\qquad$ $f(b) - f(a)$ $\qquad$ $f(a)$



Average rate of change of f with respect to x =

Average rate of change, version 2

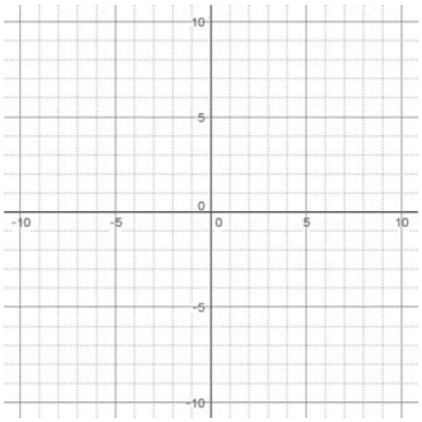
What do the labels in the diagram (and the average rate of change formula) become if we replace the label b on the x asis by $a + h$?



Average rate of change of f with respect to x =

Absolutely Vital Functions 2: Exponential and Logarithmic Functions

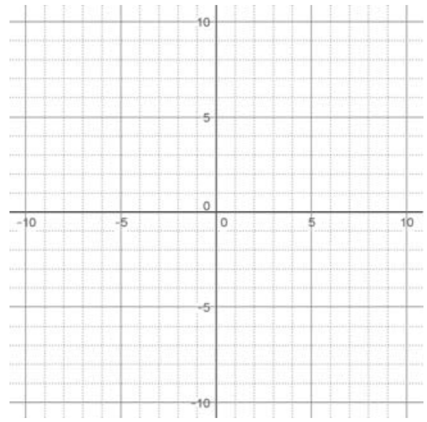
$f(x) = b^x$ Choose any $b > 1$ (4 pts. required)



domain: range:

equation of asymptote:

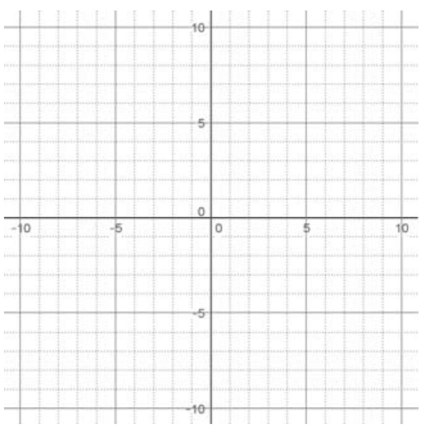
$f(x) = \left(\frac{1}{b}\right)^x$ Choose any $b > 1$ (4 pts. required)



domain: range:

equation of asymptote:

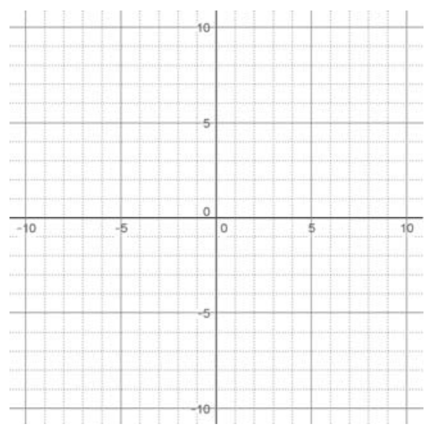
$f(x) = \log_b x$ Choose any $b > 1$ (4 pts. required)



domain: range:

equation of asymptote:

$f(x) = \log_{1/b} x$ Choose any $b > 1$ (4 pts. required)



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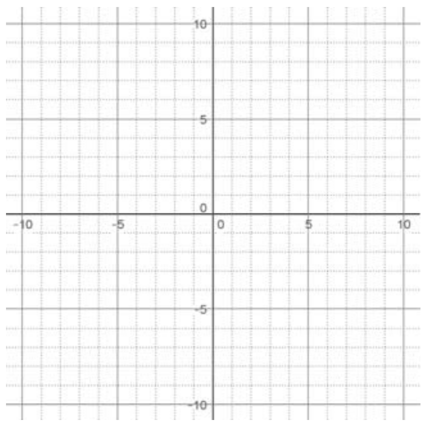
equation of asymptote:

Absolutely Vital Functions 2: Exponential and Logarithmic Functions

$f(x) = b^x$ Choose any $b > 1$ (4 pts. required)



domain: range:

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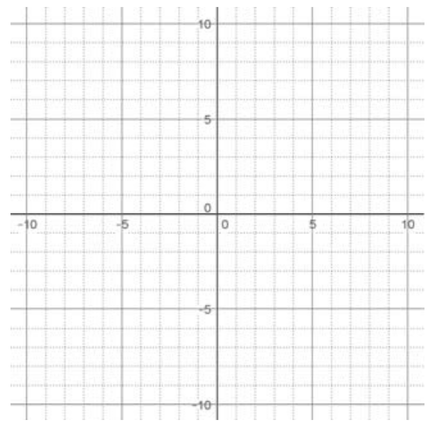
$f(x) = \left(\frac{1}{b}\right)^x$ Choose any $b > 1$ (4 pts. required)



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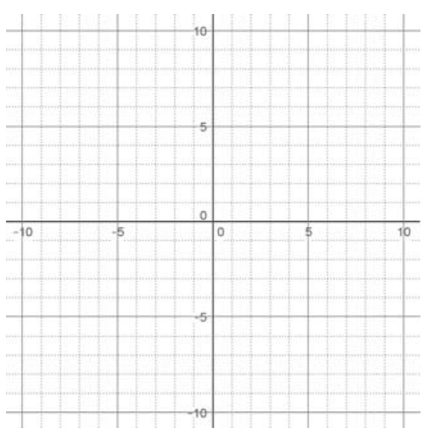

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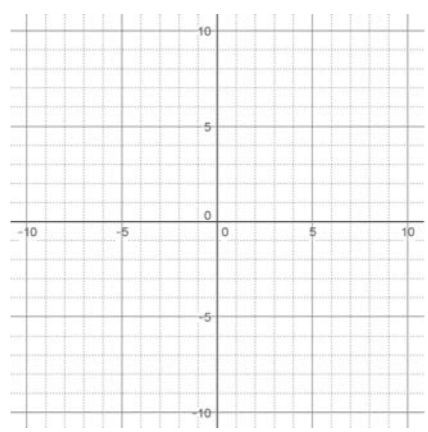


domain: range:

equation of asymptote:

Polynomials **Name:**

1. Graph $y = x^n$ for $n = 1, 3, 5, 7$ on your calculator with a window range of [-1,1] in both the x and y direction. State what happens to the graph of the function as n becomes a larger odd number.

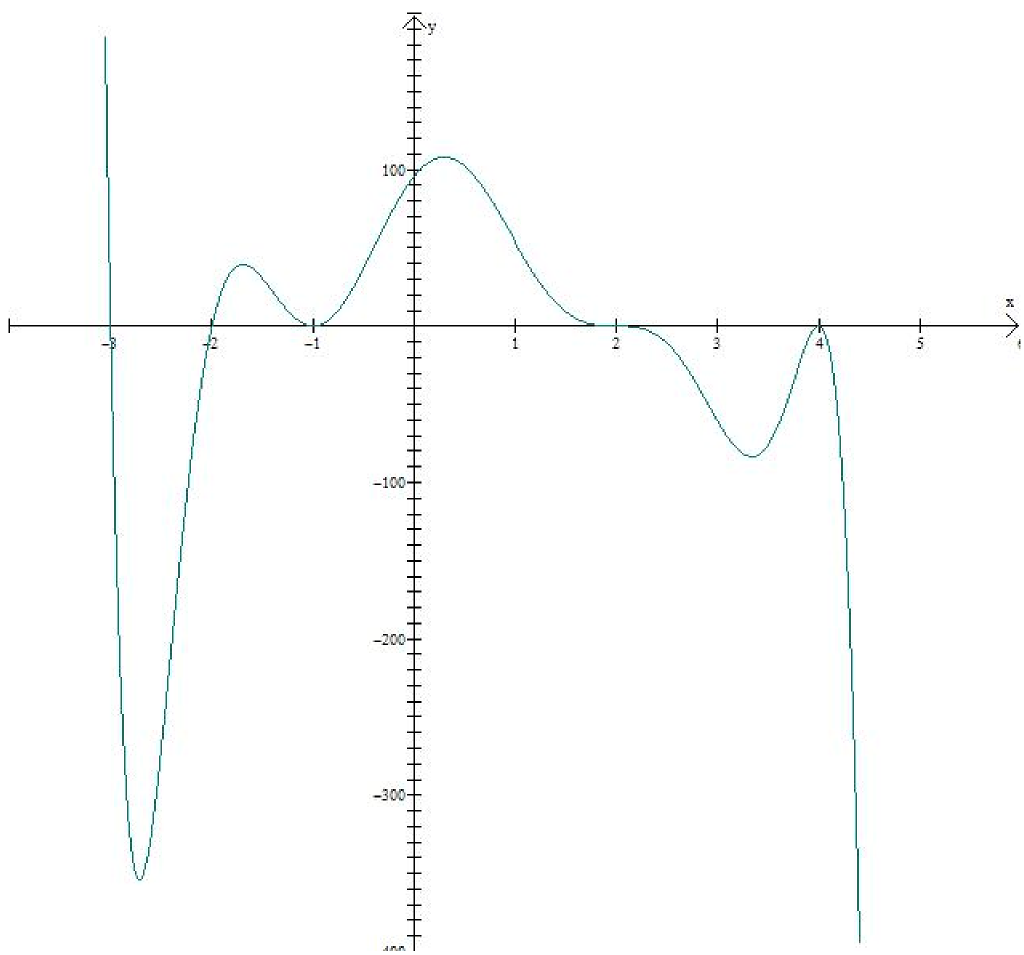
2. Graph $y = x^n$ for $n = 2, 4, 6, 8$ on your calculator in the same window as above. State the pattern.

3. Graph $y = -(x-3)(x-4)(x+2)$ on your calculator. Indicate the x and y intercepts of your graph.

4. Graph $y = (x-3)(x-3.1)(x+2)$ on your calculator, and then explain what you think $y = (x-3)^2(x+2)$ should look like based on your graph.

5. Use the patterns obtained above to graph $y = (x-2)^2(x+4)(x-5)^3(x+1)^2(x-6)(x+7)^2$. Evaluate x and y intercepts. Make sure your graph details all the important features of the graph near the x-axis.

6. Determine a (factored) polynomial given by the graph shown below, given that the y-intercept is 96.



Ok, so we can graph factored polynomials. But what if they're not factored?

7. Divide $P(x) = 3x^3 + 2x^2 - 5x + 1$ by $x - 2$ using long division and indicate your remainder. Compare with the value of $P(2)$ and speculate.

8. Prove your speculation. That is, divide $P(x) = 3x^3 + 2x^2 - 5x + 1$ by $x - k$ using long division. (You must be very careful for this part to work out properly.)

9. What you discovered above is called the factor theorem. Use it to show that $x - c$ is a factor of $(x - b)^3 + (b - c)^3 + (c - x)^3$.

10. Is $x - 1$ or $x + 1$ a factor of $x^{100} - 4x^{99} + 3$?

11. For a given polynomial with real coefficients, for any real numbers a and b if $a + \sqrt{b}$ is a root, then so is its conjugate. Thus we can answer the following question: construct a polynomial whose five roots include 3, $2i$ and $1 - \sqrt{3}$. Do not give a factored answer.

12. Use the calculator to help you find *all* the complex roots of $f(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$. (Hint: find the real roots with the calculator, then divide those roots away. Use the quadratic formula (or factoring) to find the remaining roots.)

13. Using the same $f(x)$ from the previous problem, determine the roots of $-f(-x)$.

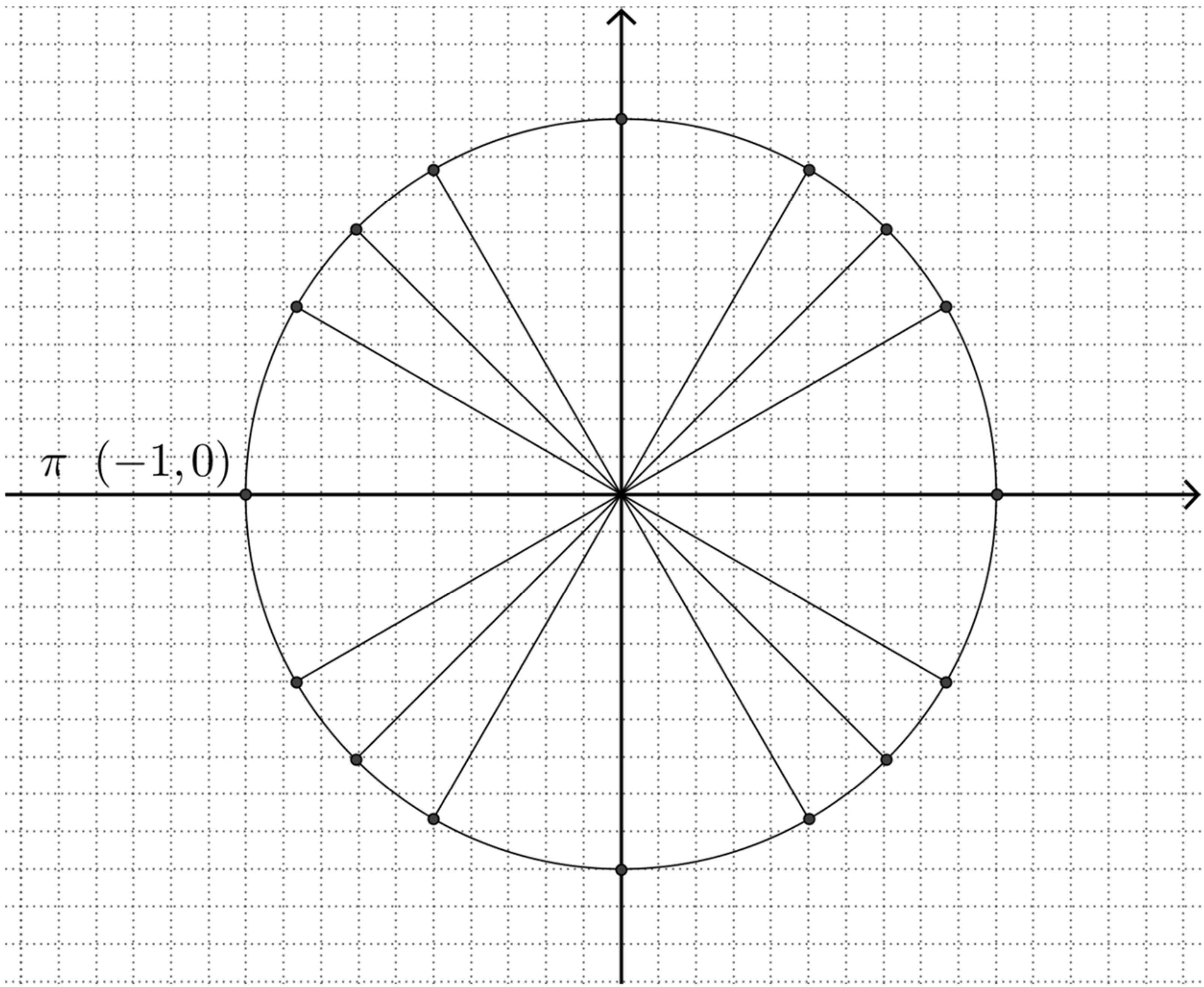
14. A rectangular piece of cardboard with perimeter 30 inches is folded by two equally spaced creases that are parallel to one of the sides of the cardboard. The result makes a prism with open ends that are equilateral triangles. State the volume formula based on x, the length of one of the sides of the equilateral base, determine the value of x that maximizes the volume, and state the maximum volume.

15. A box is to be formed from a rectangular piece of cardboard of dimensions 22 by 32 centimeters. Small squares of equal size are to be cut from the corners of this piece and then the resulting flaps will be folded up to create a box with no lid. State an appropriate volume formula based on the size of the cut squares, determine the size of the squares that maximizes the volume, and state that maximum volume.

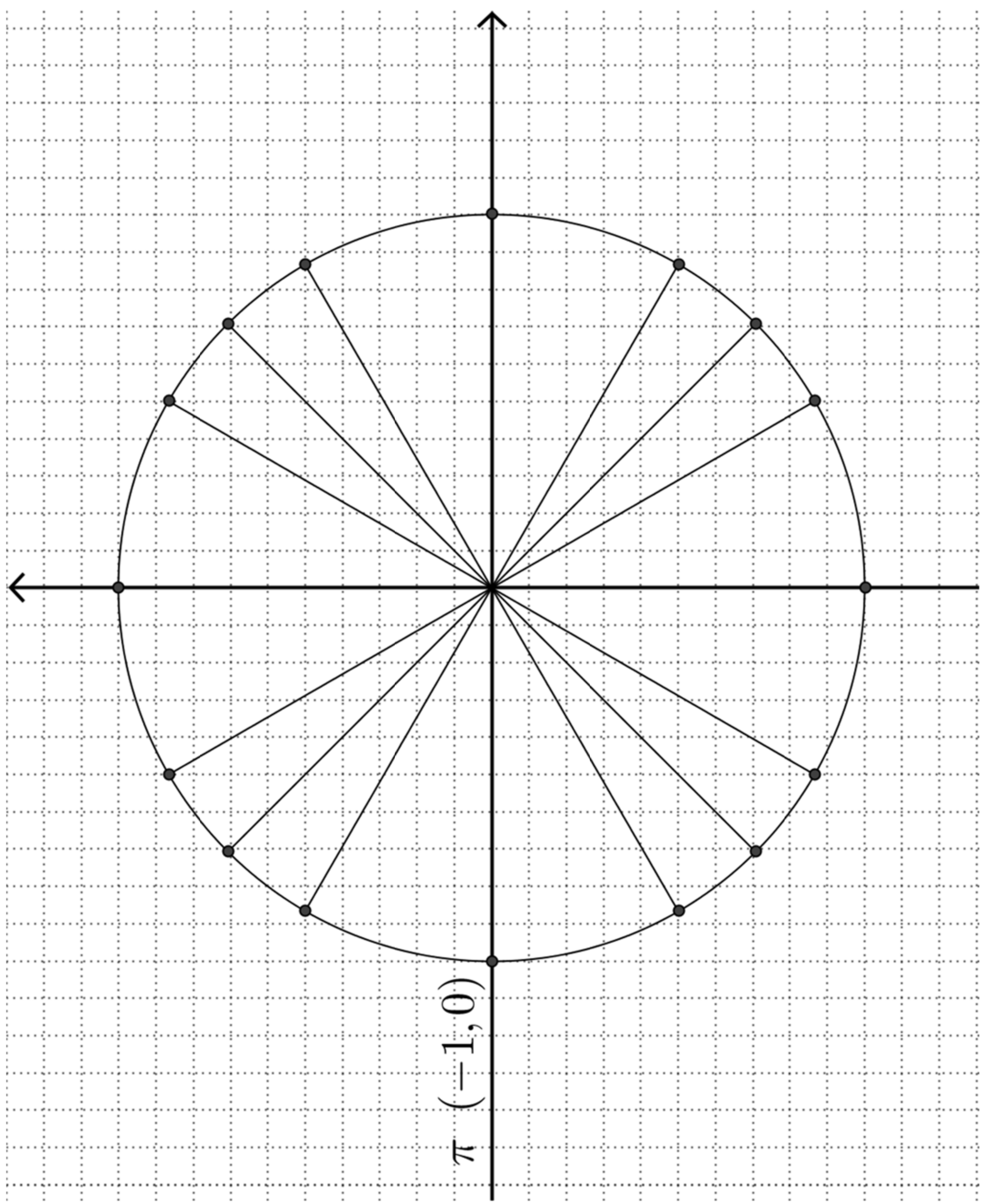
16. Repeat the above problem except make the box have a lid (ultimately, this means cutting a C-shaped piece off of the original rectangle).

$g(x)$	domain	range	intercepts	eqs. of asymptotes	end behavior	$g^{-1}(x)$
$-(x+1)^2+9$						
-5						
$5x-2x^3$						
$\frac{-4}{(x+2)^3}$						
$2(x-1)(x-5)$						
$2+\sqrt{-x}$						
$3\sqrt{x+4}$						
$\frac{x^2-3x-4}{2x^2-8x}$		skip this				skip this
3^{x+1}						
$-\left(\frac{1}{2}\right)^x+1$						
$4\log_3(-x)$						

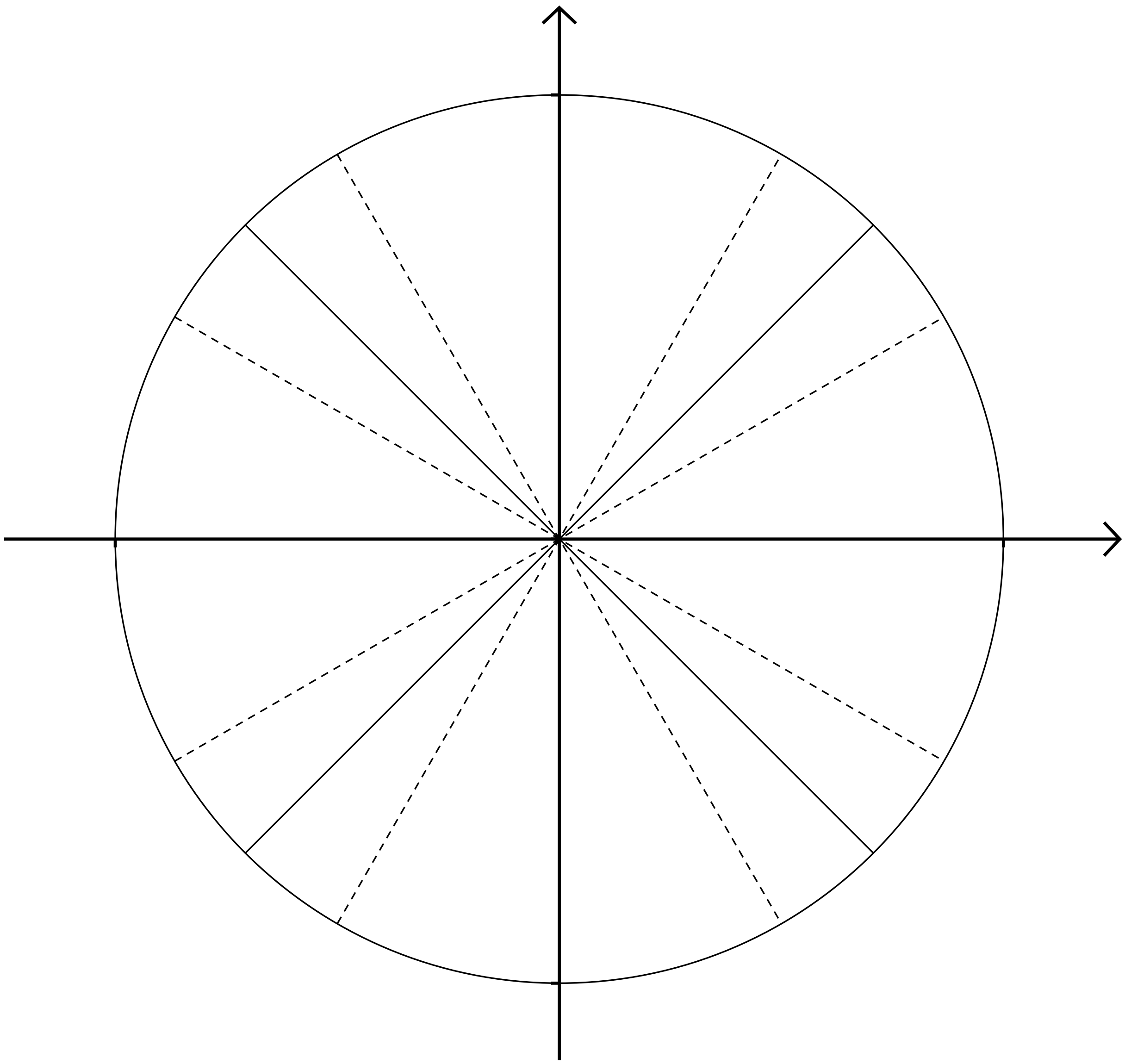
Name ____________________________________

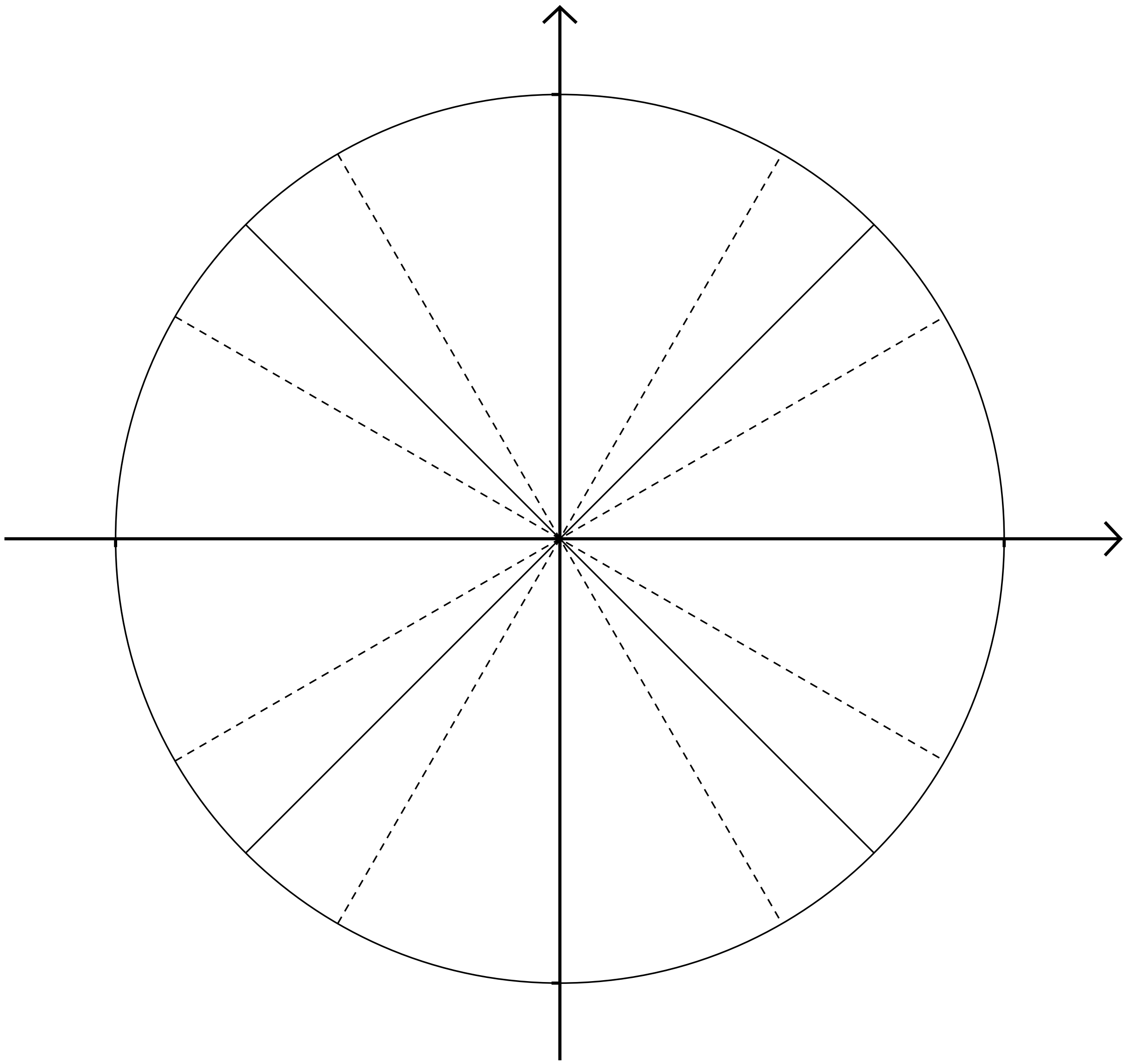


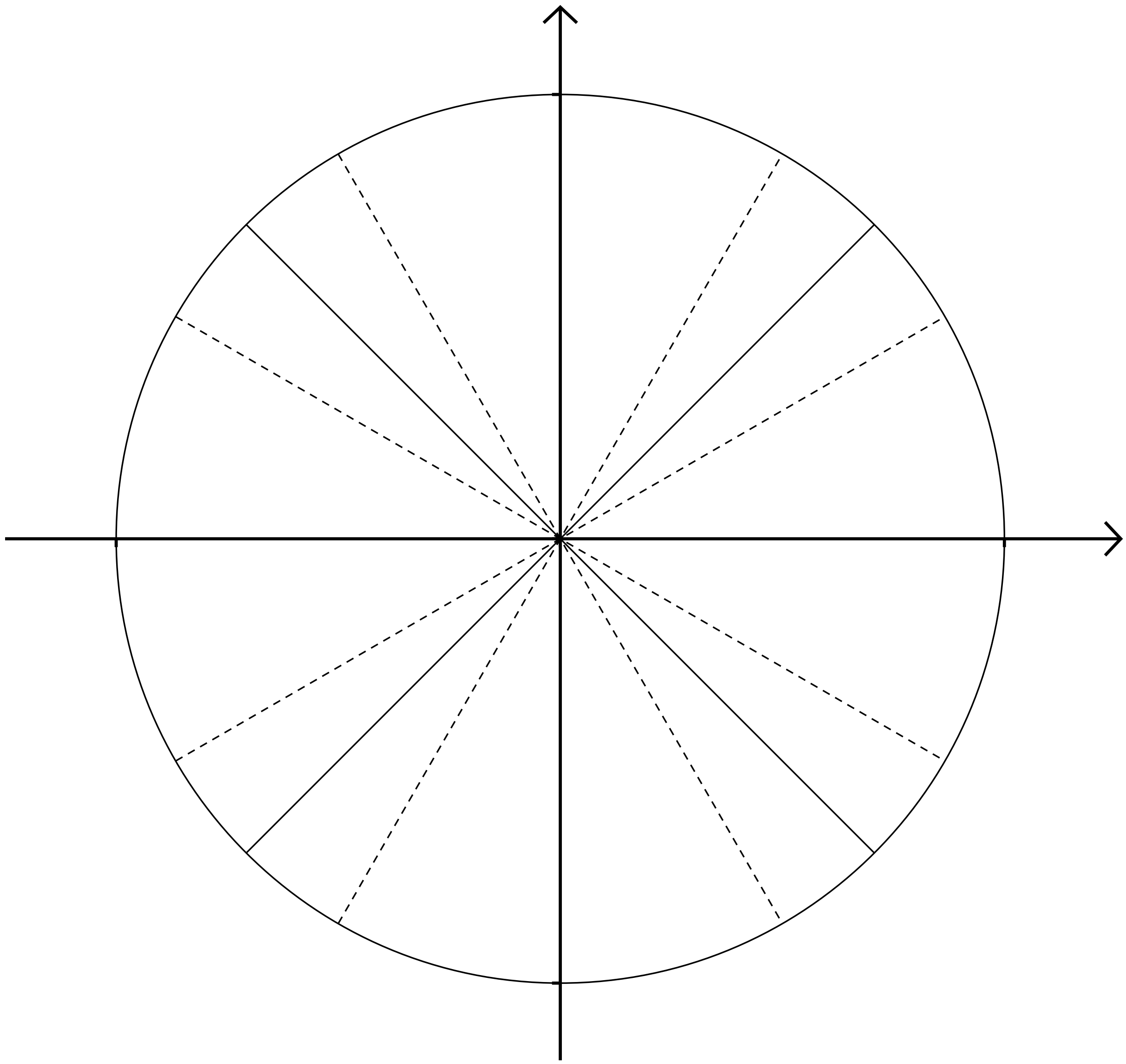
Name ______________

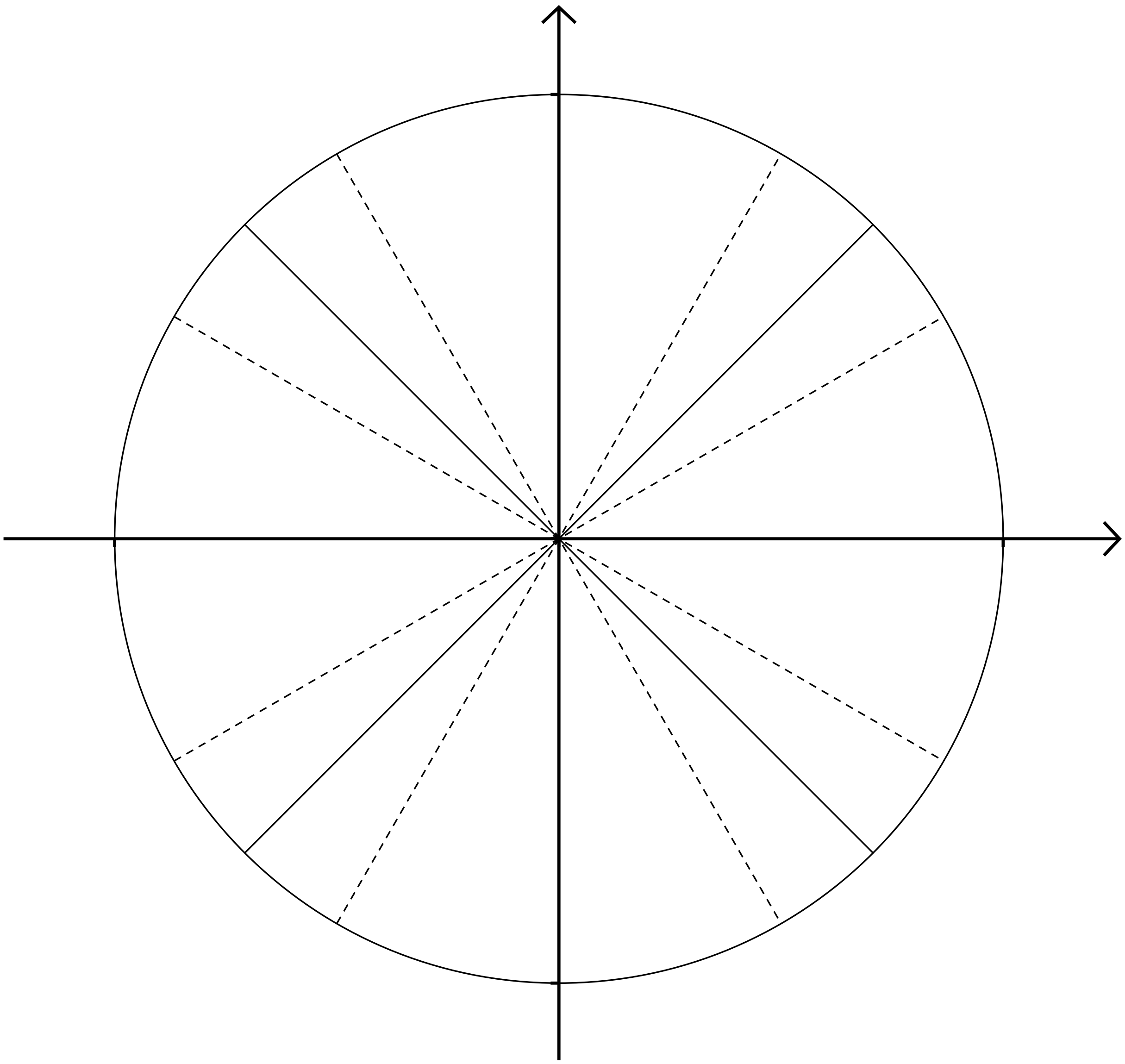


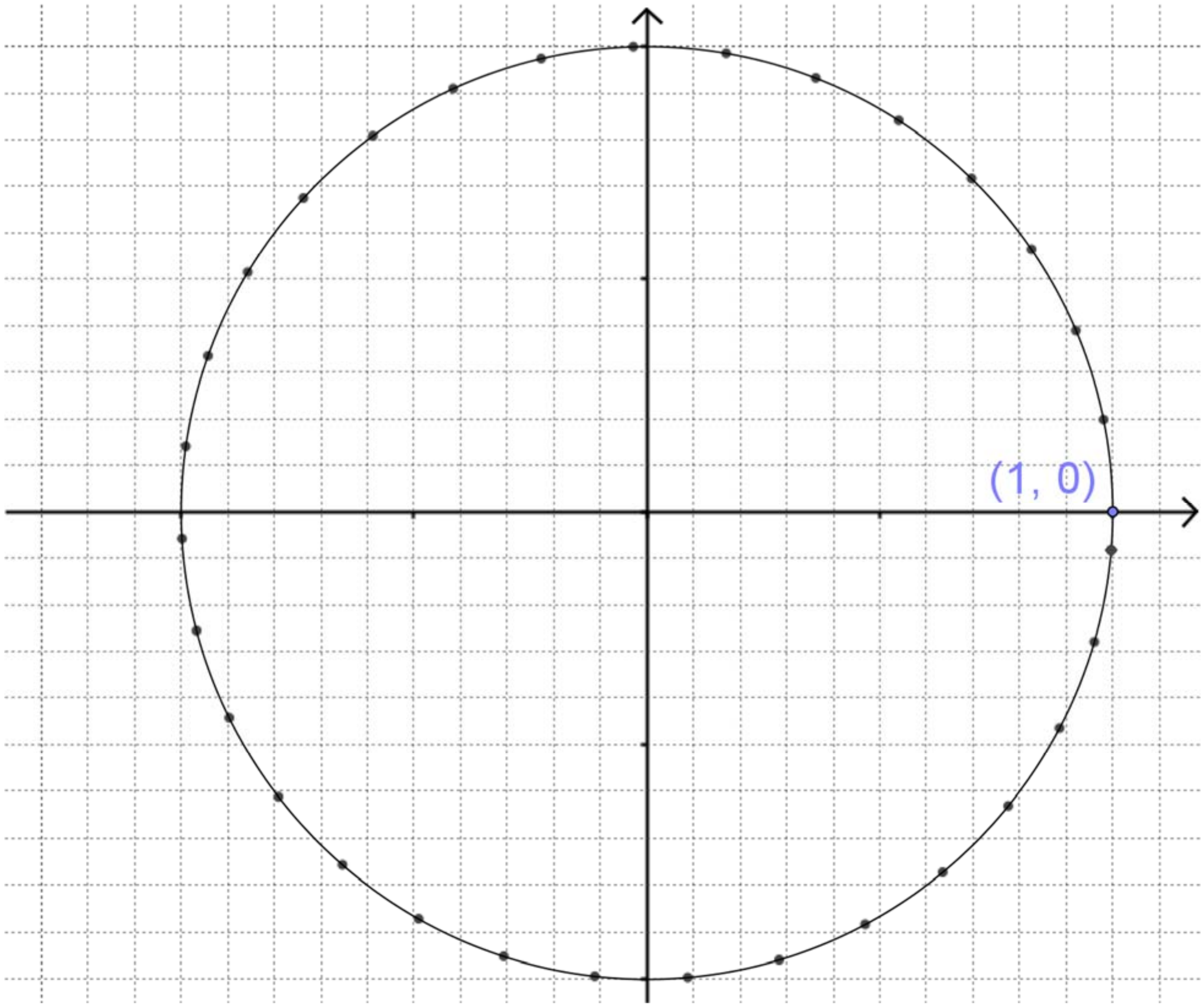
Section A.8 Blank Unit Circles







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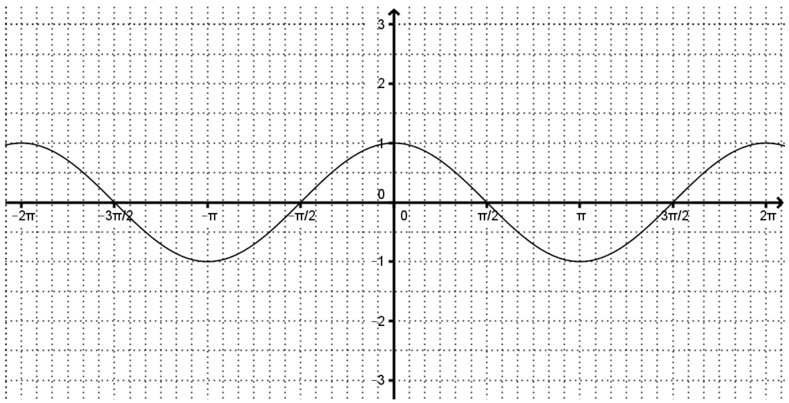
Graphs of all six trigonometric functions

Name ______________________________

Use the graphs of $y = \cos x$ and $y = \sin x$ to help you graph the other four trig functions. Be sure to graph carefully the points associated with special inputs where the output is equal to $0, \pm\frac{1}{2}, \pm 1$, or ± 2. Sketch dotted asymptotes where they exist as well. It is usually easiest to begin with the zeros and asymptotes and then to mark where the function values will be positive and where they will be negative. Finally, determine the domain and range of function.

The graph of $y = \cos x$ is shown.
Graph $y = \sec x$



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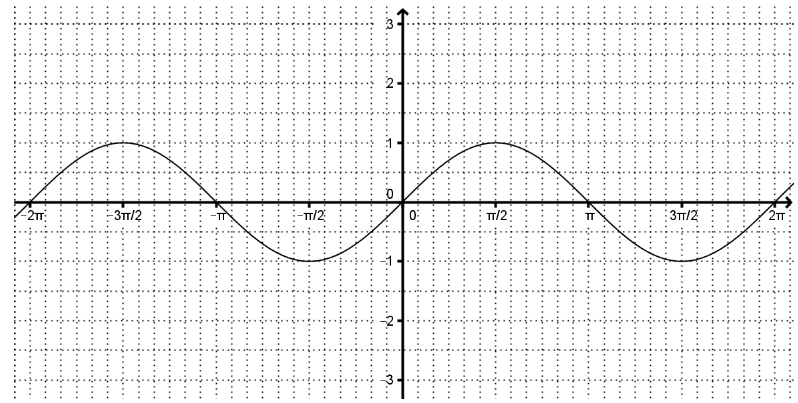
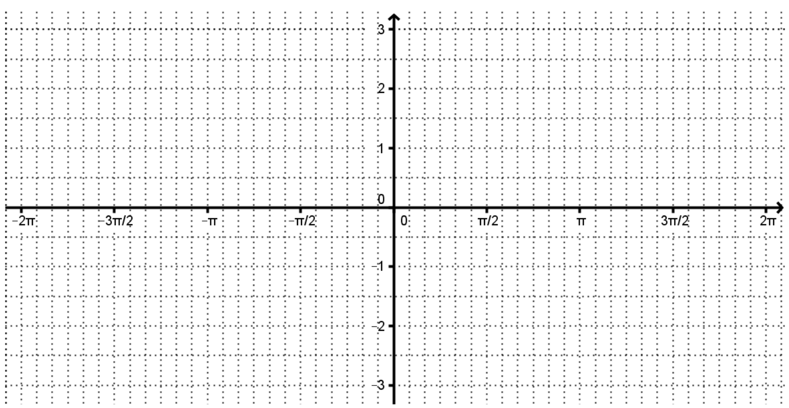
The graph of $y = \sin x$ is shown.
Graph $y = \csc x$



domain: range:

Graph $y = \tan x$



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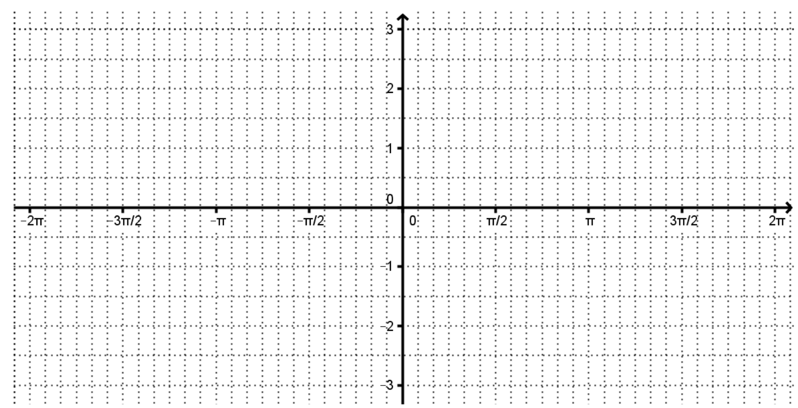
Graph $y = \cot x$



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Graphs of all six trigonometric functions

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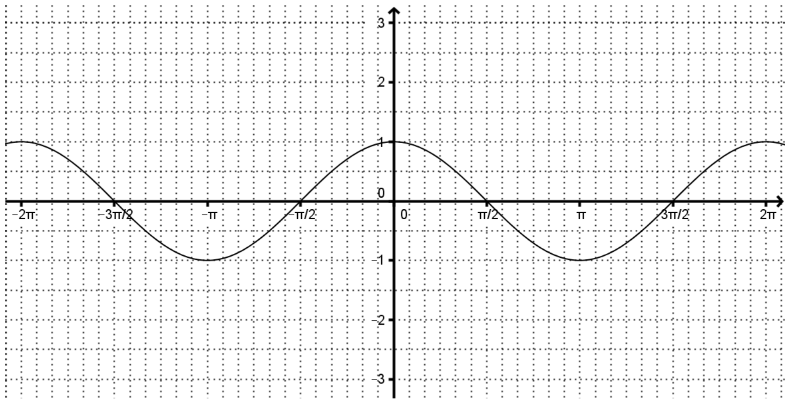
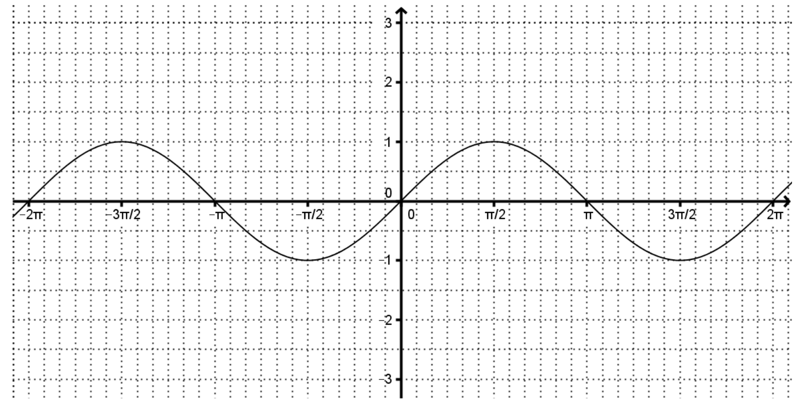


domain: range:

The graph of $y = \sin x$ is shown.
Graph $y = \csc x$



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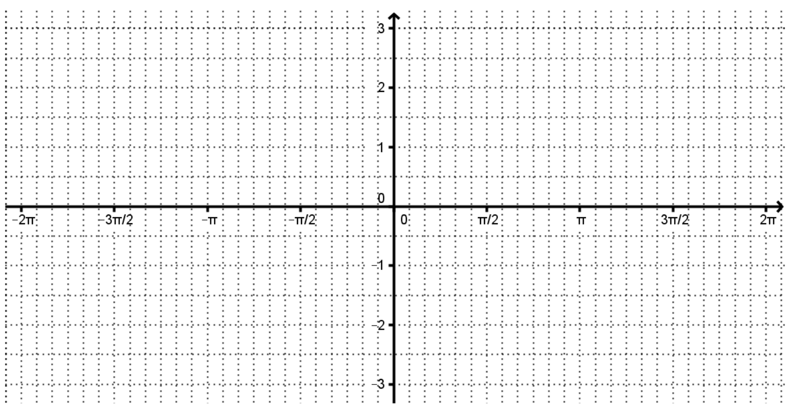
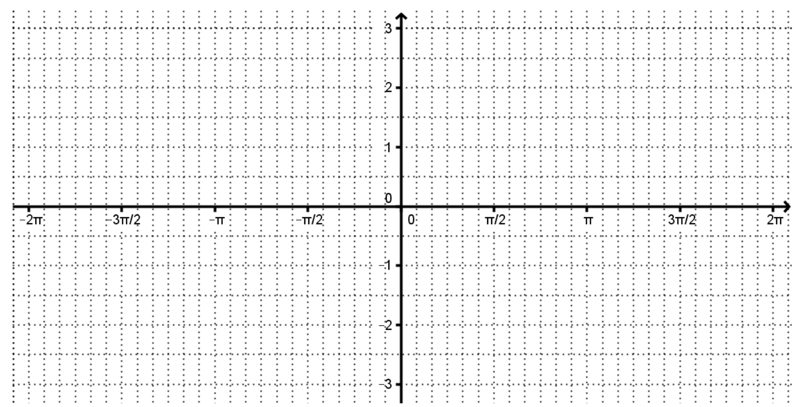
Graph $y = \tan x$



domain: range:

Graph $y = \cot x$



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Graphs of all six trigonometric functions

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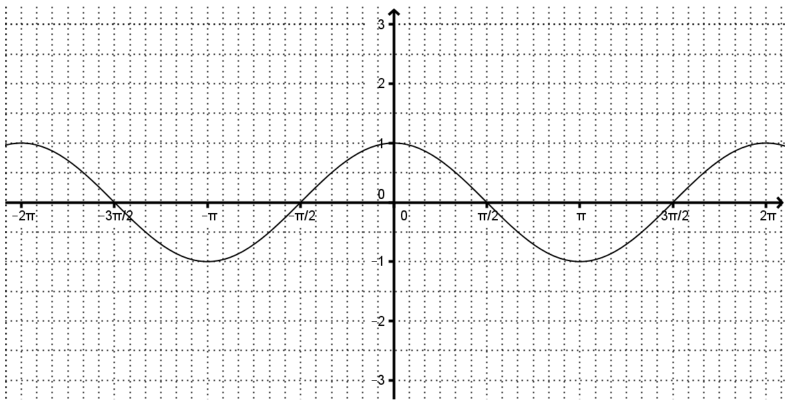

domain: range:

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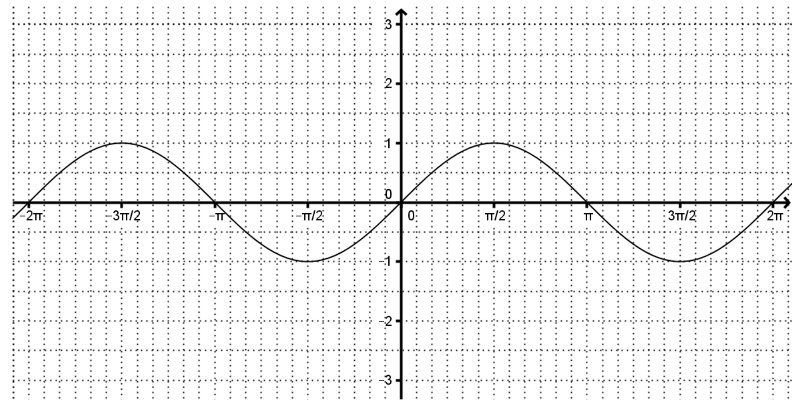
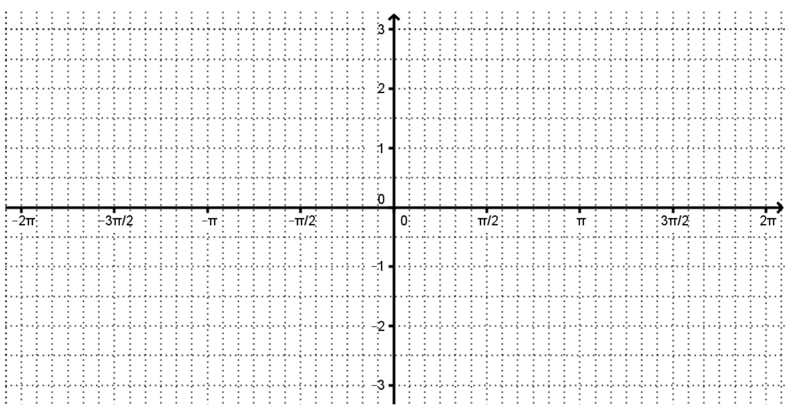
Graph $y = \csc x$



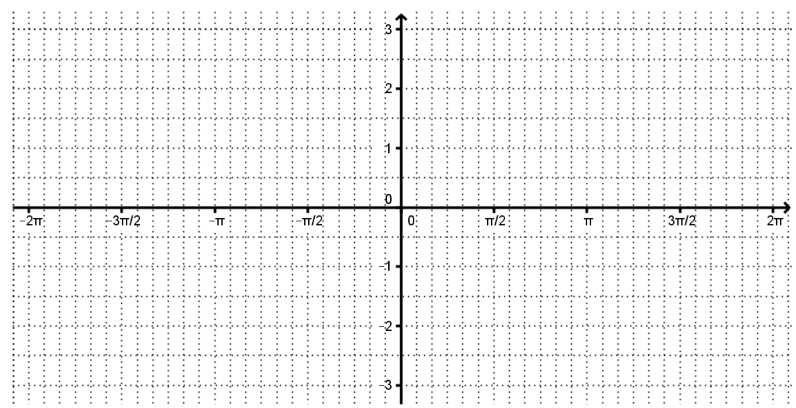
domain: range:

Graph $y = \tan x$



domain: range:

Graph $y = \cot x$



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Graphs of all six trigonometric functions

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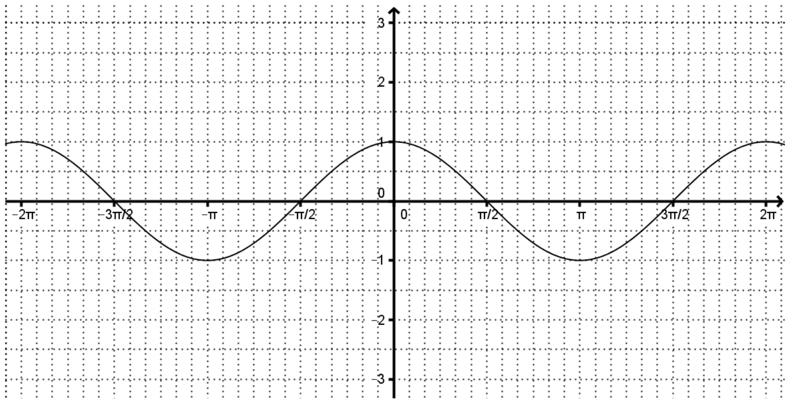

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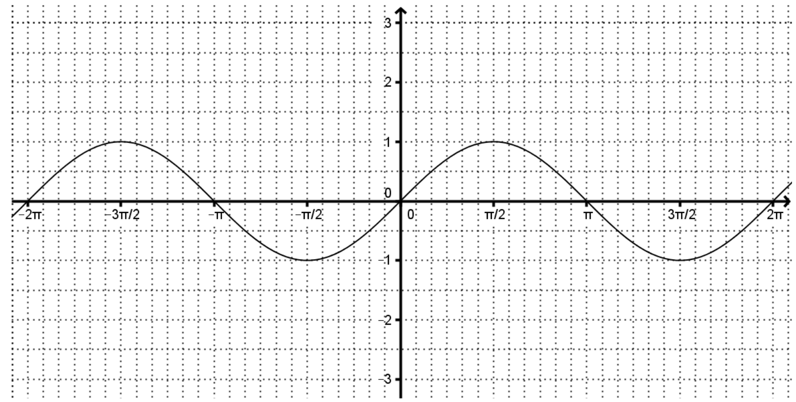
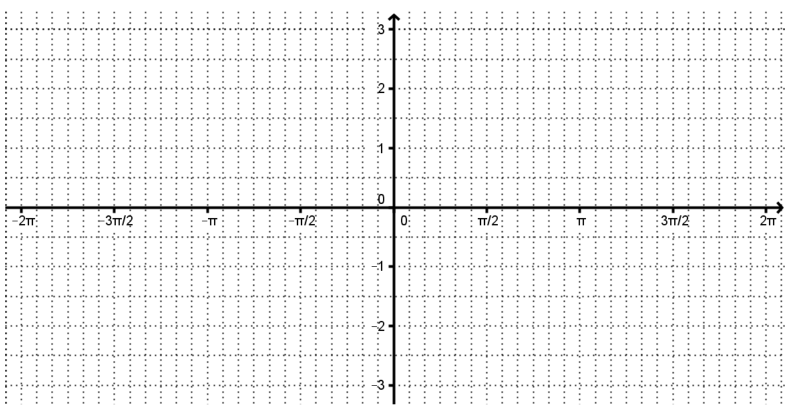
Graph $y = \csc x$



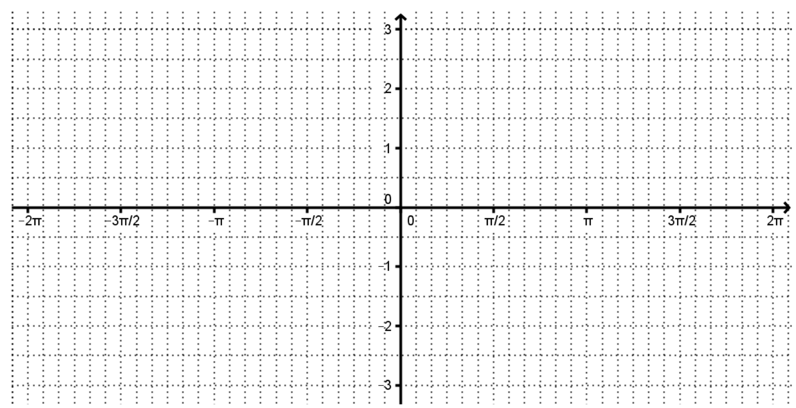
domain: range:

Graph $y = \tan x$



domain: range:

Graph $y = \cot x$



domain: range:

Composition of Trig and Inverse Trig Functions

1. $\text{Sin}^{-1}(\sin \pi/3)$
2. $\text{Sin}^{-1}(\sin -\pi/4)$
3. $\text{Sin}^{-1}(\sin \pi/12)$
4. $\text{Sin}^{-1}(\sin 4\pi/3)$
5. $\text{Sin}^{-1}(\sin 5\pi/3)$
6. $\text{Sin}^{-1}(\sin 7\pi/12)$
7. $\sin(\text{Sin}^{-1} 1)$
8. $\sin(\text{Sin}^{-1} ½)$
9. $\sin(\text{Sin}^{-1} 7)$
10. $\sin(\text{Sin}^{-1} 1/7)$
11. $\text{Cos}^{-1}(\cos \pi/3)$
12. $\text{Cos}^{-1}(\cos -\pi/4)$
13. $\text{Cos}^{-1}(\cos 2\pi/3)$
14. $\text{Cos}^{-1}(\cos 4\pi/3)$
15. $\text{Cos}^{-1}(\cos \pi/8)$
16. $\text{Cos}^{-1}(\cos 5\pi)$
17. $\cos(\text{Cos}^{-1} 1)$
18. $\cos(\text{Cos}^{-1} ½)$
19. $\cos(\text{Cos}^{-1} 3/5)$
20. $\cos(\text{Cos}^{-1} 10)$
21. $\text{Tan}^{-1}(\tan \pi/3)$
22. $\text{Tan}^{-1}(\tan -\pi/4)$
23. $\text{Tan}^{-1}(\tan \pi/7)$
24. $\text{Tan}^{-1}(\tan 4\pi/3)$
25. $\text{Tan}^{-1}(\tan 5\pi/3)$
26. $\text{Tan}^{-1}(\tan \pi/2)$
27. $\text{Tan}^{-1}(\tan 7\pi)$
28. $\tan(\text{Tan}^{-1} 1)$
29. $\tan(\text{Tan}^{-1} 100)$
30. $\text{Sin}^{-1}(\sin 7\pi/8)$
31. $\text{Cos}^{-1}(\cos -2\pi/3)$
32. $\sin(\text{Sin}^{-1} 11)$
33. $\cos(\text{Cos}^{-1} 7/15)$
34. $\tan(\text{Tan}^{-1} 72.6)$
35. $\sin(\text{Sin}^{-1} 72.6)$
36. $\text{Sin}^{-1}(\sin 9\pi/4)$
37. $\text{Cos}^{-1}(\cos -\pi/3)$
38. $\text{Tan}^{-1}(\tan 3\pi/4)$
39. $\text{Sin}^{-1}(\sin 16\pi/15)$
40. $\text{Cos}^{-1}(\cos 13\pi/7)$

Answers

1. $\pi/3$	2. $-\pi/4$	3. $\pi/12$	4. $-\pi/3$	5. $-\pi/3$
6. $5\pi/12$	7. 1	8. ½	9. undef.	10. 1/7
11. $\pi/3$	12. $3\pi/4$	13. $2\pi/3$	14. $2\pi/3$	15. $\pi/8$
16. π	17. 1	18. ½	19. 3/5	20. undef.
21. $\pi/3$	22. $-\pi/4$	23. $\pi/7$	24. $\pi/3$	25. $-\pi/3$
26. undef.	27. 0	28. 1	29. 100	30. $\pi/8$
31. $2\pi/3$	32. undef.	33. 7/15	34. 72.6	35. undef.
36. $\pi/4$	37. $2\pi/3$	38. $-\pi/4$	39. $-\pi/15$	40. $\pi/7$

Solving Triangles

For each of the following sets of 3 pieces of information, at least one triangle can be formed. Tell how many triangles are possible, make sketches of the possible triangle(s), and find the missing sides and angles. (If two or more triangles are possible, give two triangles that work.) Measurements should be correct to two places after the decimal.

1. ΔFAN : $\angle F = 35°$, $\angle A = 120°$, $n = 7$

2. ΔFUN : $f = 11$, $u = 3$, $n = 9$

3. ΔTRY : $\angle T = 95°$, $t = 15$, $r = 4$

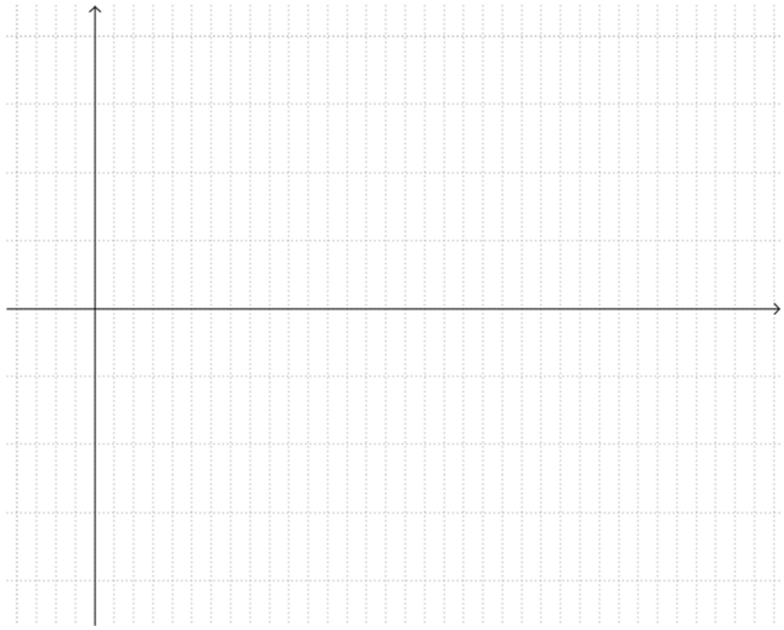
4. ΔFRY : $f = 7$, $\angle Y = 37°$, $y = 6$

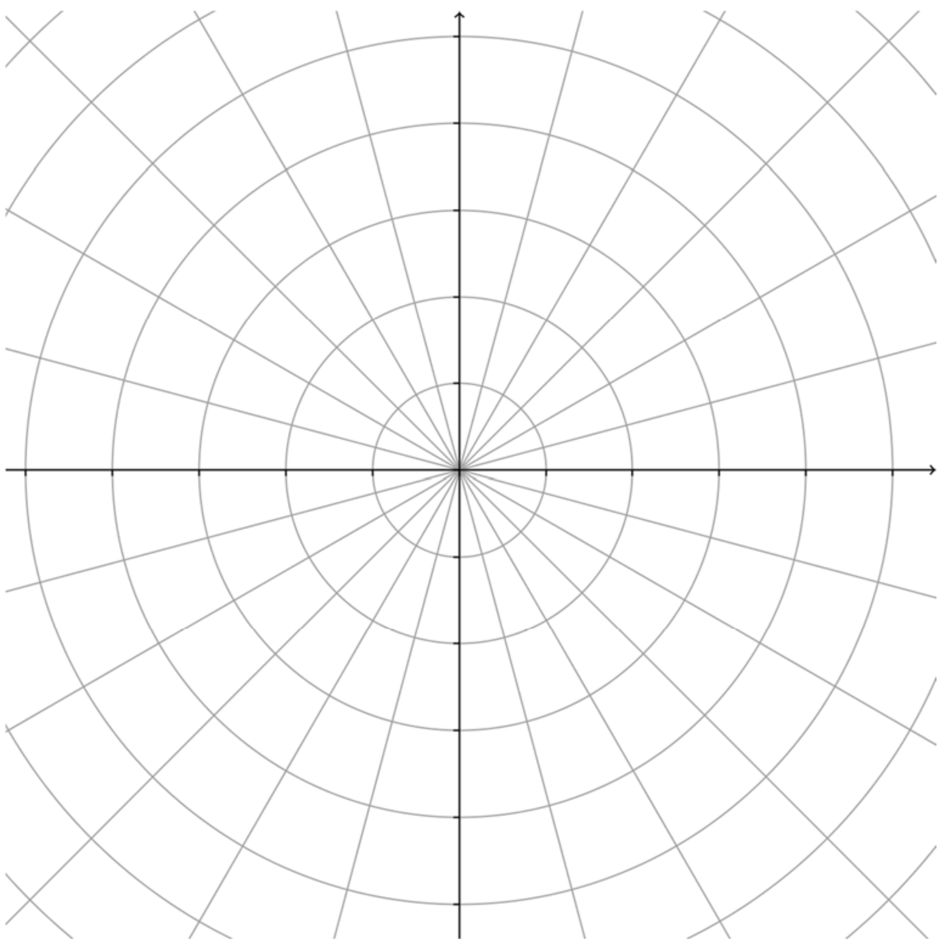
5. ΔCRY : $c = 7$, $\angle Y = 37°$, $y = 8$

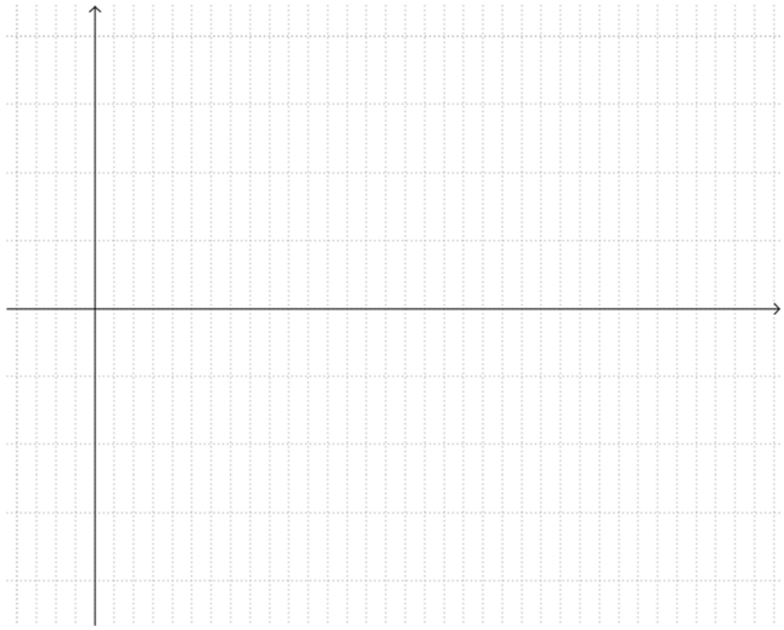
6. ΔSPY : $s = 5$, $p = 11$, $\angle Y = 52°$

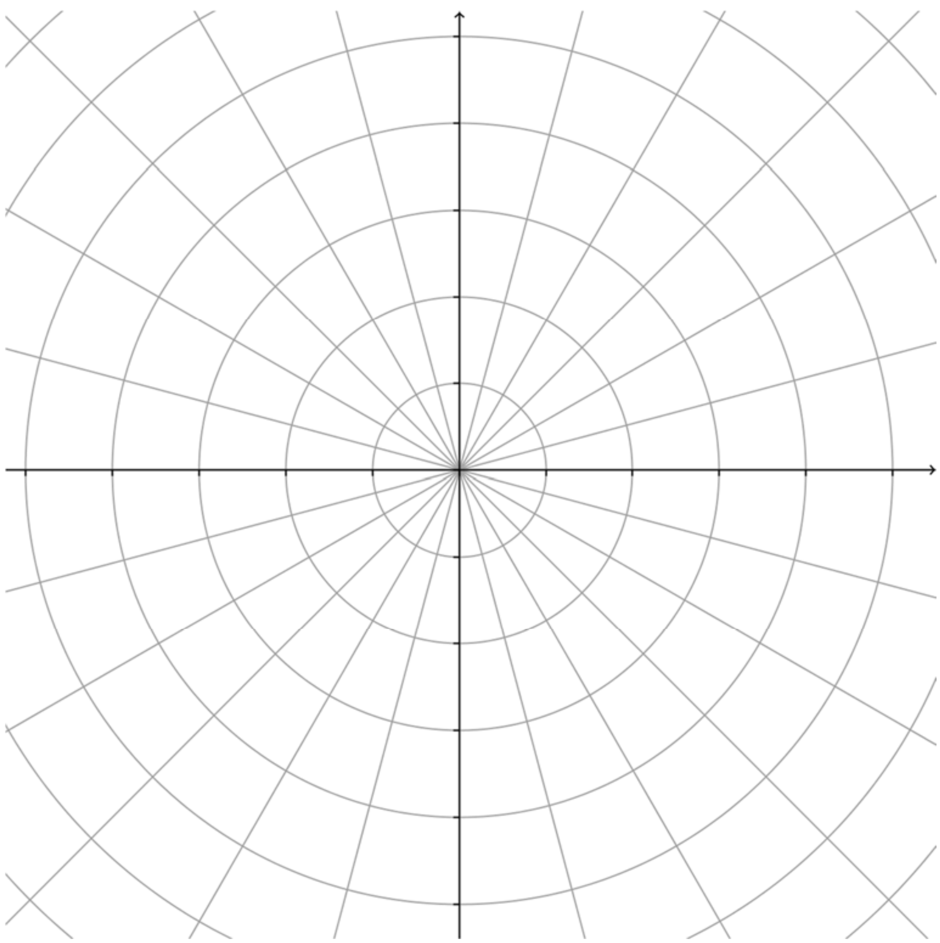
7. ΔZIG : $\angle Z = 48°$, $\angle I = 22°$, $z = 10$

8. ΔZAG : $\angle Z = 48°$, $\angle A = 22°$, $\angle G = 110°$









Probability Distributions **Name:**

Probability distributions tell us the probability that a variable will take on a certain value or range of values.

1. The binomial distribution is used when there are *repeated, independent* trials with *two* possible outcomes (success or failure), and the probability of success is the same in each trial. State and evaluate the expression that represents the probability of rolling 4 sixes on 14 normal, 6-sided dice.

   Press 2nd, Vars and find the functions *binomcdf* and *binompdf*. Entering binompdf(n,p,k) will return the probability of exactly k successes out of n attempts, if on each attempt the probability of success is p. Entering binomcdf(n,p,k) will return the probability of *at most* k successes out of n attempts, if on each attempt the probability of success is p.

2. What is the probability of rolling at most 4 sixes on 14 normal, 6-sided dice? Then, use sigma notation to show the mathematical expression you would evaluate.

3. What is the probability of getting at least 7 heads when 12 normal two-sided coins are tossed?

4. For a binomial distribution of n trials with probability of success p per trial, the mean value for the number of successes is np, that is, $\mu = np$. The standard deviation is given by the equation $\sigma = \sqrt{np(1-p)}$.A binomially distributed experiment with 16 trials has a standard deviation of 1.6. What is the probability of at least 5 successes for this experiment? What is the average number of successes for this experiment? (Is there more than one answer possible?)

5. The Poisson distribution is used when we consider infrequent events. Given an expected number λ (the Greek letter "lambda") of occurrences, we can compute the probability of k occurrences according the formula $p(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$. Compute the probability of 5 accidents in a month at a construction site if, on average, there are 2 in one month.

6. Find poissonpdf(λ,k) and poissoncdf(λ,k) on the calculator. They operate in a similar manner to their binomial counterparts. What is the probability of 7 failures on an assembly line in a day, if $\lambda = 10$?

7. What is the probability of at least 7 failures on an assembly line in a day, if $\lambda = 10$?

8. What is the domain of $p(k)$? At what value(s) of k is $p(k)$ a maximum, for a given λ?

9. Since λ is the expected rate of occurrences, it should make sense that the mean of a Poisson distribution is λ. It is also true that the standard deviation for a Poisson distribution is $\sqrt{\lambda}$. Calculate the probability of 3 defects from an assembly line in one day, if the standard deviation of the number of defects is equal to 2.5.

Lots of Statistics (and a Few Probability) Problems **Name:**

1. You're stocking the fireplace with logs from a pile. The pile consists of two kinds of logs, both of which have normally distributed burn times. The first type of log has a mean burn time of 20 minutes and a standard deviation of 3 minutes. The second type of log has a mean burn time of 15 minutes and a standard deviation of 5 minutes. There are twice as many of the first type of log in the pile as the second type of log. A log is randomly selected from the pile and is observed to burn for at least 23 minutes. What is the probability that the log selected is of the second type of log?

2. Adjust the previous question by changing the observation to say that the log burns for 23 minutes, with an error of $\pm .5$ minutes. Now what is the probability that the log selected is of the second type?

3. The number 25! ends in how many zeroes?

4. Which is the better test score with respect to the other scores from the same test: a 97 on a test where $\overline{x} = 55$ and $\sigma = 16$ or an 83 on a test where $\overline{x} = 68$ and $\sigma = 6$? Why?

5. Consider the data set 4, 5, 6, 6, 7, 7, 7, 9, 12. Calculate the mean, median, mode, interquartile range, and standard deviation.

6. Draw a boxplot representing the data set above.

7. The removal of which data point would affect the mean the most? The median? The mode? The standard deviation?

8. What happens to the standard deviation if each data point is increased by 5? Multiplied by 3?

9. A normal, six-sided die is rolled repeatedly until a 1 appears. What is the probability that the first 1 appears on the nth roll?

10. Using the previous problem, express as a series the expected number of rolls for a 1 to appear on a normal die.

11. Suppose 95% of the sneakers manufactured by a shoe company have no defects. In order to find the 5% that do have defects, inspectors carefully look over every pair of sneakers. Still, the inspectors aren't perfect either, and 8% of the defective sneakers pass inspection, and 1% of the good pairs fail the inspection test. What is the probability that a pair of sneakers has a defect if it passes inspection?

12. Suppose you play a game in which 5 coins are tossed simultaneously. If 1, 2, 3, or 4 heads occur, you win \$1 for each head. If all heads or all tails occur, you lose \$20. What is the game's expected payoff?

13. Prove by induction on n that $||\vec{u_1} + \vec{u_2} + \ldots + \vec{u_n}|| \leq ||\vec{u_1}|| + ||\vec{u_2}|| + \ldots + ||\vec{u_n}||$ for all positive integers n.

14. Sarina and Joel are playing a game with sets of numbers. Sarina selects a number from the set of integers $\{1, 2, \ldots, 5\}$, and Joel selects a number from the set of integers $\{1, 2, \ldots, 10\}$. What is the probability that Joel's number is larger than Sarina's?

15. What score on an IQ test is at the 80th percentile if the scores are normally distributed about a mean of 100 and a standard deviation of 10?

16. Out of 68 randomly selected George School people, 29 of them are in favor of turning Red Square into a bumper car rink. Construct a 95% confidence interval for p, the proportion of the entire George School population in favor of turning Red Square into a bumper car rink.

17. How many randomly selected people would you need to ask a survey question in order to be 95% confident that the actual proportion who would respond *yes* to the survey question is within 0.01 of the sample proportion?

18. Consider the following data that represents years of education: 8, 10, 12, 12, 13, 14, 16, 16, 18, 19.
Also consider the following data that represents income in thousands: 23, 24, 32, 32, 26, 35, 54, 63, 45, 43.
Use linear regression to get a model for the data above. How strong would you say the correlation is between education and income?

19. Give an addition pair of data that would lower the correlation between education and income. Why does that point lower the correlation?

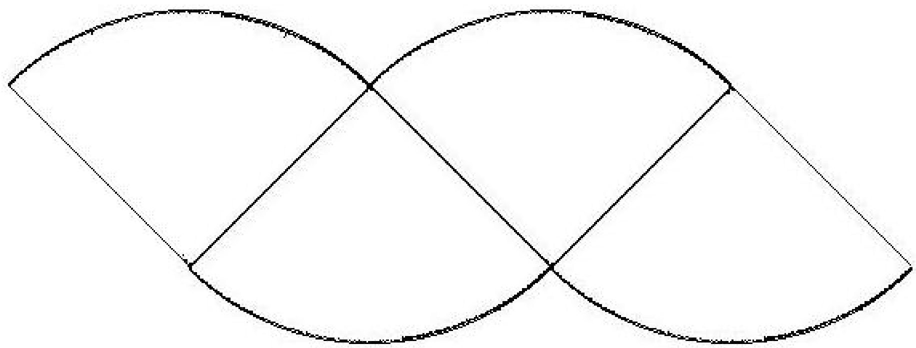
Limit Arguments **Name:**

Answer the questions or fill in blanks as indicated.

Recall that the definition of π is that is the ratio of a circle's circumference to its diameter. From this definition, we are able to come up with a formula for the circumference of a circle based on its radius. Provide that formula.

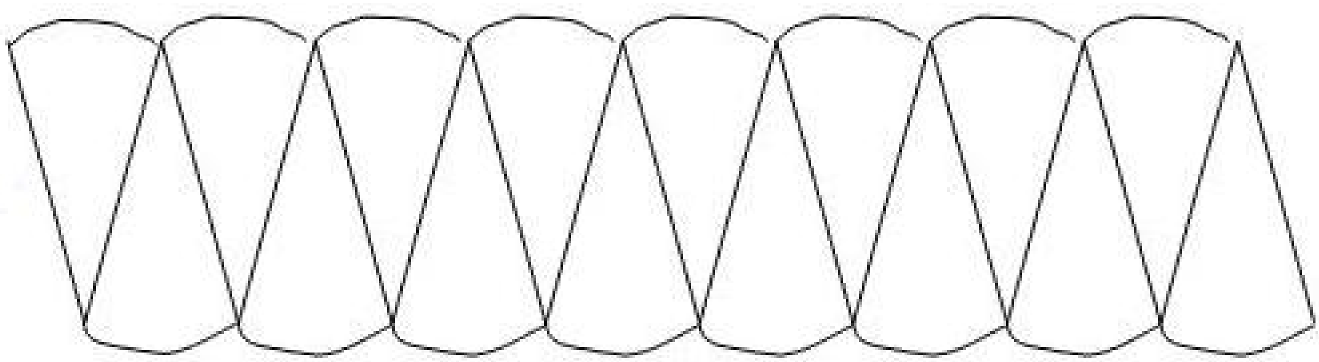
You may also recall from geometry that the area of a circle can also be determined from its radius. That formula is $A = \pi r^2$. We will work through a short limit argument that explains why this formula is correct.

First, let's consider cutting a circle into four equal pieces, as though the circle was a pizza. Now, we want to rearrange the slices to approximate a rectangle.



Since half the circumference is on the bottom of the picture, the "length" of this rectangle is ________. The radius acts as the "width" of this rectangle.

This is a pretty silly looking rectangle, but we can get a better approximation by cutting the circle into sixteen pieces and rearranging those pieces instead.



What is the "length" and "width" of this rectangle?

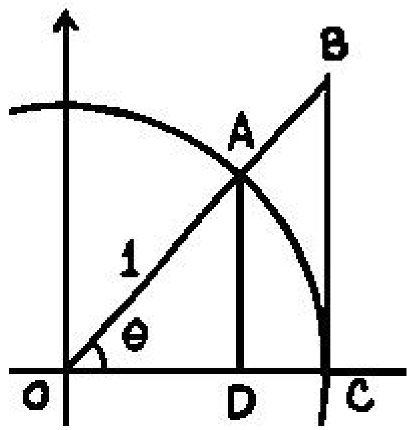
As we create more and more slices of our circle, hopefully you can see that the shape made when the slices are rearranged becomes more and more like a rectangle. Let n equal the number of slices of the circle. Let $A(n)$ represent the area of the shape created by rearranging the n slices in the way depicted above. Write a sequence of mathematical symbols, using notation we've learned in class, to state what is happening.

The idea of infinitely small pieces summing together to yield a finite result is one of the two main ideas at the heart of calculus. You may have already seen this at work when you've analyzed geometric sequences such as the sequence 4, 2, 1, .5, .25, ... What is the common ratio of this sequence?

Recall (or learn) that the formula for the sum, S_n, of the first n terms of a geometric sequences is $S_n = \frac{a_1(1-r^n)}{1-r}$, where a_1 represents the first term of that sequence and r represents the common ratio. If $|r| < 1$, evaluate $\lim_{n\to\infty} S_n$. (Hint: What happens to a number between 0 and 1 when you raise it to a large power?)

The other main idea is that the limit of the ratio of two infinitely small (or large) things can also yield a finite result. We know, for example, that $\lim_{\theta \to 0} \sin \theta = 0$ and $\lim_{\theta \to 0} \theta = 0$ (or we better know, anyway). But what about the ratio of these two functions?

In the figure below, the circular arc has a radius of 1 and the angle θ is such that $0 \leq \theta \leq \frac{\pi}{2}$ (90°).



Find the areas of right triangle OAD, sector OAC, and right triangle OBC (use trigonometry to help). Note that OA and OC are radii, and that the area A of a sector is given by $A = \frac{1}{2}r^2\theta$.

Thus, conclude that $(\cos \theta)(\sin \theta) \leq \theta \leq \dfrac{\sin \theta}{\cos \theta}$ by using your area expressions above.

Divide the inequality chain above by $\sin \theta$ and find the limit as θ approaches 0 of each of the three parts of the inequality.

Use your work to conclude that $\lim_{\theta \to 0} \dfrac{\theta}{\sin \theta} = \lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$. (This result turns out to be important later.)

A Big HL Review **Name:**

Don't be intimidated by anything. No, really.

1. Solve for k if $4\ln 2 - 3\ln 4 = -\ln k$.
2. Solve for x if $\log_3(x+17) - 2 = \log_3(2x)$.
3. Find the three cube roots of $8i$. Express each in $a + bi$ form.
4. Suppose $z = (b+i)^2$, where b is a positive, real number. If $z = r\text{cis }60°$, find b.
5. Solve for x if $\sin 2x = \sqrt{2}\cos x$, on the interval $0 \le x \le \pi$.
6. Suppose obtuse angle B is such that $\tan B = 5/12$. Evaluate $\sin 2B$ and $\cos 2B$.
7. If $2\tan^2\theta - 5\sec\theta = 10$, evaluate $\sec\theta$.
8. If $A = \begin{bmatrix} 1 & 2 \\ k & -1 \end{bmatrix}$ and $A^2 = \mathbf{0}$ (the zero matrix, a matrix with all zeroes), find k.
9. Prove that $5^n + 9^n + 2$ is divisible by 4 for all positive integers n by induction.
10. Find the complex roots of $z^2 + 2z + 4 = 0$ and convert the answers to trigonometric form.
11. The sides of a triangle are $x-2$, x, and $x+2$ and the largest angle is 120°. Find x. What's the area of the triangle?
12. A bag contains 4 red and n green balls. If two balls are chosen without replacement, the probability of two red balls is 2/15. Find the value of n.
13. Evaluate without a calculator: $\sin\left(\tan^{-1}\frac{12}{5} + \arctan\frac{3}{4}\right)$.
14. Sketch the graph of $y = \arctan(x)$.
15. Graph $y = 3^{x-5} + 2$ and state its inverse.
16. Euler's Formula (an equation most easily proved by series in calculus) states that $re^{i\theta} = r(\cos\theta + i\sin\theta)$. Write i in the form $re^{i\theta}$, then raise that expression to the ith power and simplify it to determine the exact value of i^i.
17. Newton's law of cooling states that if an object initially at temperature T_0 is placed in a location whose average temperature is T_R, the object's temperature will decrease according to the function $C(t) = T_R + (T_0 - T_R)e^{-kt}$, where k is a positive constant and t is measured in units of time.

    It's 2 A.M. and you're a CSI investigating a murder at a hotel room. Suppose you're told that the body was discovered at midnight ($t = 0$) and its temperature was 85°F. The temperature of the room has been kept constant at 65°F. If the temperature of the body is currently 80°F, find the time of death. (Hint: Use 98.6°F for the temperature of a living body.)
18. Use the factor theorem and synthetic division to assist in solving the equation $0 = x^4 + 2x^3 - 7x^2 - 8x + 12$.
19. How many 4 digit numbers have at least one 7 or at least one 9?
20. Solve the system of equations $3x - y\sqrt{3} - 2z = -3$, $e^2x + 2y - \pi z = 4$, and $-2x + y + z = 1$ for x, y, and z.
21. In quadrilateral $ABCD$, $AB = 17$, $BC = 62$, $CD = 101$, $\angle B = 40°$, and $\angle C = 75°$. Find the area of the quadrilateral.
22. State the domain and range of $f(x) = \log_2(\log_3(\log_4 x))$.
23. Solve for x if $e^{\sin x} = 2x^2 - 3x$.
24. Prove that $(1 + \cot^2 x)(1 - \cos 2x) = 2$ is an identity.
25. Find the smallest positive x-intercept of $f(x) = -.8 + 2^{\sin 2x}$.
26. Show that $\cos 3x = 4\cos^3 x - 3\cos x$.
27. Solve for θ, if $0 \le \theta < 2\pi$: $\sin 2\theta = \sin\theta$.

28. If $\cos\theta = 4/5$ and θ is in the first quadrant, give exact values for $\sin 2\theta$ and $\sin(\theta/2)$.

29. Sketch the graph of $y = 4\sin(2x + \pi/2) - 3$. Indicate amplitude, period, horizontal shift, vertical shift.

30. Evaluate $\tan 345°$.

31. Prove that $\dfrac{\sin\theta\sec\theta}{\tan\theta + \cot\theta} = \cos^2\theta - \cos 2\theta$.

32. Prove that $a^{\log b} = b^{\log a}$.

33. Suppose $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. Evaluate $\cos(a - b)$.

34. Let $T = \begin{bmatrix} x & x-2 & 5 \\ x+2 & x & -3 \\ x+2 & x & x \end{bmatrix}$. If $|T| = 0$, solve for x.

35. If $\log(xy^3) = \log(x^2y) = 1$, determine the value of $\log xy$.

36. Find the exact value of $\cos(\sin^{-1}\frac{5}{13} - \cos^{-1}\frac{4}{5})$.

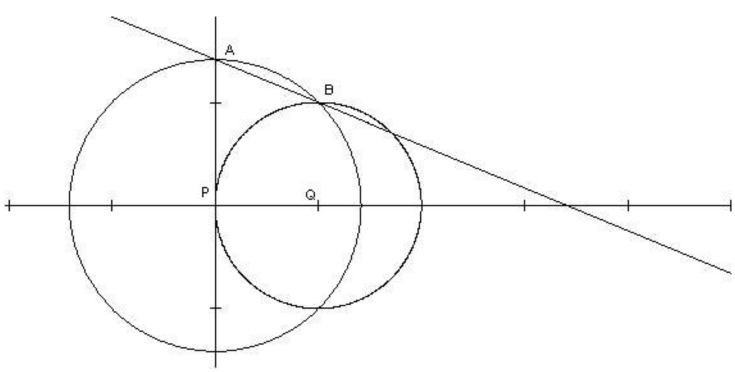
37. Suppose 95% of the sneakers manufactured by a shoe company have no defects. In order to find the 5% that do have defects, inspectors carefully look over every pair of sneakers. Still, the inspectors aren't perfect either, and 8% of the defective sneakers pass inspection, and 1% of the good pairs fail the inspection test. What is the probability that a pair of sneakers has a defect if it passes inspection?

38. Box A contains 6 red balls and 2 green balls. Box B contains 4 red balls and 3 green balls. A fair cubical die with faces numbered 1 through 6 is thrown. If an even number is rolled, a ball from Box A is selected. If an odd number is rolled, a ball from Box B is selected.
a. Calculate the probability that the ball selected is red.
b. Given that the ball selected is red, calculate the probability it came from box B.

39. Evaluate $\displaystyle\sum_{n=0}^{100} i^n \sin\left(n\frac{\pi}{2}\right)$. (Remember, $i = \sqrt{-1}$.)

40. Solve for x if $-\tan x = \tan 2x$ on the interval $0 \le x < 2\pi$.

41. This set of questions concerns two circles. Circle P is centered at the origin and has a *variable* radius r. Circle Q has radius 1 and is centered at the point $(1, 0)$. A line is drawn from the point $A(0, r)$ on P through the intersection B of the two circles in the first quadrant. See picture.



a. Find the coordinates of B in terms of r. Recall that the formula for a circle of radius d centered at (h, k) is given by the formula $(x - h)^2 + (y - k)^2 = d^2$.
b. Find an equation for the line through A and B.
c. Find a *simplified* expression for the x-intercept of the line through A and B in terms of r.
d. What is the limiting value of the x-coordinate of the x-intercept from part c as r approaches 0?

EC. Suppose $P(r, \theta)$ is any point on the circle with center $C(a, \alpha)$ and radius R. Find the polar equation of the circle in terms of r, θ, a, α, and R. (Hint: Find an appropriate triangle to make.)

Another Big Review
Practice problems, go!

Part 1: With Calculator

1. A normal, six-sided die is rolled. Whatever the result is, that many coins are flipped following the die roll. State and evaluate an expression using sigma notation that represents the expected number of heads.

2. A fairly safe intersection has an average of 2.3 accidents per year. What is the probability that at least 4 accidents occur in a given year at this intersection?

3. There are 12 members of the Retford Liberation Front. How many ways could they select a group of 5 people to storm the second floor of the building?

4. There are about 56,302 people sitting in a local stadium watching a mini-Olympics. 32,591 people are interested in the decathlon, and 17,484 people are interested in buying a hot dog. If there are 9,842 people who are neither interested in the decathlon nor interested in buying a hot dog, how many people are interested in the decathlon but do not want to buy a hot dog?

5. Write a transformation matrix that takes an image, scales it by a factor of 2 and rotates it 30 degrees.

6. Given $(1,-1)$, $(2,-6)$, $(3,11)$, $(4,98)$, find the values of a, b, c, and d such that the graph of the function $f(x) = ax^3 + bx^2 + cx + d$ goes through all four points.

7. A tram car has been knocked off a (horizontal) bridge by Doc Ock, but Spiderman saves the day by shooting two strands, S_1 and S_2, from the tram car to the bridge to stop its fall. Spiderman is on the tram at point T, 80 feet below point P on the bridge. The two strands are connected to the bridge 210 feet apart at points A and B such that $AP = 60$ feet and $BP = 150$ feet. If the weight of the tram car is 10000 pounds, find the tension on each strand.

8. Explain induction. What is the logic that allows it to work?

9. Sketch the graph of $r = 1 + 2\cos\theta$, where $0 \le \theta \le 2\pi$. Indicate intercepts.

10. Sketch the graph of $x = 3\cos 3t$ and $y = 3\sin 4t$, where $0 \le t \le 2\pi$.

11. In $\triangle XYZ$, $\angle X = 21.1°$, $x = 6$, and $y = 9$. Find the measures of all remaining sides and angles of the triangle.

12. An infinite geometric series has a sum of 12 and its first term is r^3, where r is the common ratio. Find r.

13. Graph $y = \dfrac{x^3+1}{x^2-1}$. Indicate all asymptotes, discontinuities, and intercepts.

14. State and simplify y' if $y = \dfrac{x^3+1}{x^2-1}$. Then use that equation to determine at which x value(s) the local minimum and local maximum of y occur. (Do not just use your calculator.)

15. Describe a process that would allow you to sketch the graph of $y = \dfrac{1}{f^{-1}(x)}$, given a picture of $y = f(x)$.

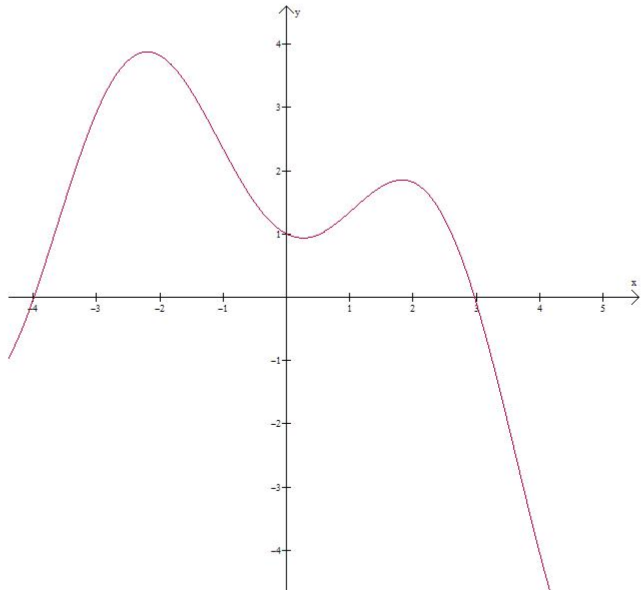
16. Solve $\log_5 x + \log_5(x-4) = 1$.

17. Evaluate $\displaystyle\sum_{n=3}^{1003} i^n$.

18. Prove that $\cot(B) + \tan(B) = 2\csc(2B)$ is an identity.

19. Solve $2\sin^2 x - \sin x = 2 - \csc x$.

20. Given the graph of y below, sketch on the same axes the graph of y'.



21. Find the third derivatives of $y = x^{11}$, $y = 15x$, and $y = 1.5x^{2/3}$.

Part 2: No Calculator

22. Evaluate $(1+i)^{12}$. Write your answer in $a + bi$ form.

23. Evaluate $\sec(345°)$.

24. Sketch the graph of $y = \sin(\pi/3 - x) + 1$. Indicate period, shifts, and amplitude.

25. Evaluate $\lim\limits_{n\to\infty} \dfrac{n^2 + 2n + 1}{n^3 - 8}$.

26. For what value(s) of K is $x = 3$ a solution of $x^3 - 5x^2 + Kx - 12 = 0$? If $x = 3$ is a solution, then what are the others?

27. Sketch the graph of $2x^2 + 8y^2 = 72$. Indicate any interesting features.

28. Evaluate $\lim\limits_{x\to-\infty} \dfrac{x-2}{\sqrt{4x^2+5}}$. Explain your reasoning.

29. Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(4, 0)$. (Hint: recall that if the square of a positive expression is minimized, then that positive expression is also minimized.)

30. Sketch the graph of $y = x(x-1)^2(x+1)^3(x-7)^3$. Indicate intercepts.

31. Let $\phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$, the golden ratio. Let $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Thus, a_n is the Fibonacci sequence. This last question guides you through some interesting properties of both these topics. You may use results from any section (whether or not you successfully proved it) to help you out in any other section.
    a. Show that $\phi - 1 = \dfrac{1}{\phi}$.
    b. Show that $\phi^{n+1} = \phi^n + \phi^{n-1}$ for all positive integers n. (Hint: Factor.)
    c. Prove by induction that $a_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$ for all positive integers n (Hint: $1 - \phi = -(\phi - 1)$.)
    d. Show that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \phi$.
    e. Show that ϕ is equivalent to the expression $\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$.
    f. Show that ϕ is equivalent to the expression $1 + \dfrac{1}{1 + \frac{1}{1+\frac{1}{1+\ldots}}}$.

B Answers to many odd-numbered problems

Chapter 1

Section 1.1 (page 12)

5. $y = -x$

7. It is not possible to write an equation for this line in slope-intercept form (though that does not mean it is impossible to write an equation for this line)

9. $y = -\frac{5}{2}x + 4$

11. A couple of the possibilities are $y + 8 = -3\left(x + \frac{1}{5}\right)$ and $y + 11 = -3\left(x - \frac{4}{5}\right)$

13. $y + 9.49 = \dfrac{5}{12}(x - 18.76)$

15. One possibility is $y - 5 = -\frac{2}{5}(x + 4)$. Another is $y - 3 = -\frac{2}{5}(x - 1)$. What's another?

17. One possibility is $y - 4 = -\frac{5}{3}(x - 10)$. Another is $y - 9 = -\frac{3}{2}(x - 7)$. What's another?

19. -32

21. $\frac{1}{100}$

23. 32

25. $\frac{4}{81}$

27. 4

29. -3

31. undefined

33. undefined

35. $\frac{1}{4}$

37. $\frac{1}{4}$

39. 10

41. -2

43. $\frac{5}{2}$

45. 2

47. $\frac{1}{3}$

49. $a = 3, b = 4$

51. $a = 2, b = 3$

53. $a = -4, b = -3$

55. $a = -2, b = -1$

57. $\frac{1}{2}\left(3 + \log_2 x + 3\log_2 y\right)$

59. $A(x) = 12x, x \geq 0$

61. $k = 4$

63. $x = 1$

67. $\dfrac{18}{x^5}$

69. $\dfrac{y}{500x^4}$

71. x-ints: $(-3, 0)$ and $(-7, 0)$ y-int: $(0, 21)$

73. x-ints: $\left(1 \pm \frac{\sqrt{2}}{2}, 0\right)$ (one root is between 1 and 2, and the other is between 0 and 1) y-int: $(0, 1)$

75. x-int: $(5, 0)$ no y-int

77. no x-int y-int: $(0, 6)$

79. x-int: $(4, 0)$ y-int: $(0, -15)$

81. x-int: $(\log_2 10, 0)$, which is between 3 and 4 y-int: $(0, -9)$

83. A is the set of all x such that x is less than or equal to 5.

85. a) $\{x \mid 1 < x \leq 3\}$
b) $\{x \mid x > -4\}$

87. a) $\{x \mid x < 2\}$
b) $\{x \mid -6 < x < -4\}$

89. $\dfrac{-(2y + 1)}{2(y + 1)}$

91. $-\dfrac{1}{4x}$

93. $-\frac{5}{x(x+h)}$

Section 1.2 (page 22)

3. $3, 1, -1, -3, -5$; $a_1 = 3$, $a_n = a_{n-1} - 2$

5. a) $\frac{4}{3}$
b) $a_1 = 7$, $a_n = a_{n-1} + \frac{4}{3}$
c) $a_n = 7 + \frac{4}{3}n - 1$
d) $\frac{97}{3}$

7. a) $-\frac{4}{3}$
b) 16
c) $a_1 = 16$, $a_n = a_{n-1} - \frac{4}{3}$
d) $a_n = 16 - \frac{4}{3}n - 1$
e) $-\frac{28}{3}$

9. explicit: $a_n = 5\left(\frac{4}{5}\right)^{n-1}$, recursive: $a_1 = 5$, $a_n = \frac{4}{5}a_{n-1}$

11. a) $\sqrt[3]{\frac{2}{3}}$

b) 18

c) $a_1 = 18$, $a_n = a_{n-1}\left(\sqrt[3]{\frac{2}{3}}\right)^{n-1}$

d) $a_n = 18\left(\sqrt[3]{\frac{2}{3}}\right)^{n-1}$

e) 6.105

13. -2488

15. One possibility is $\sum_{i=8}^{12} \frac{7}{i}$. What's another?

17. One possibility is $\sum_{i=5}^{5} \frac{\sqrt{i}}{10(i-2}$. What's another?

19. One possibility is $\sum_{i=2}^{6} \sqrt[i]{5+i}$. What's another?

21. One possibility is $\sum_{i=0}^{8}(-1)^i$. What's another?

23. 99

25. -2

27. 55

29. 15

31. $15\frac{1}{2}$

33. -7

35. 0

37. -1

39. -1

41. $1+i$

43. $1+i$

47. arithmetic, 630

49. $\frac{2}{9}$

51. 40800

53. $a_1 = 45$, $S_\infty = \frac{45}{1-\frac{2}{3}}$

55. $a_1 = \frac{1}{4}$, $S_4 = \frac{\frac{1}{4}\left(1-(\frac{3}{2})^4\right)}{1-\frac{3}{2}}$

57. $a_1 = -\frac{7}{48}$, $S_\infty = \frac{-\frac{7}{48}}{1+\frac{1}{2}}$

59. $a_1 = 0.3$, $a_2 = 0.03$, $a_3 = 0.0033$, $a_n = 0.3(0.1)^{n-1}$

61. $\frac{14}{3}$

Section 1.3 (page 29)

1. One is $y-9 = -2(x+5)$. To find another, think about how much y will change by if x changes by 1.

2. $\frac{3}{2}$

3. $\frac{1}{8}$

4. $\frac{3}{4}$

5. $-\frac{2}{3}$

6. undefined

7. -4

8. $\frac{2}{3}4$

9. $\frac{1}{4x^3}$

10. x-int: $\left(\log_3 160, 0\right)$, which is between 4 and 5, y-int: $(-159, 0)$,

11. $\frac{(x+6)(x+4)}{24(x-3)}$

12. -1

13. $1-i$

14. $15\frac{2}{5}$

15. $\frac{256i-2}{2-i}$

16.

Chapter 2

Section 2.1 (page 40)

1. domain: $(-\infty, 5\,]$ range: $(0, \infty)$

3. domain: $(-3, 7]$ range: $(-\infty, 2]$

5. $f(x) = -1$ domain: $(-\infty, \infty)$ range: $[-1, -1]$

7. a) domain: $[-4, 2)$ range: $(-3, 5]$

b) 1.2

c) -2

d) $[-4, -4] \cup [0, 2)$

e) 4.0

f) $[-4, 1.2]$

g) $[1.2, 2)$

h) $[-4, 2)$ except for 1.2

i) $[-4, 2)$

j) $[-4, 1.2]$

9. a) domain: $(-1, 5.5)$ range: $(-3, 3]$

b) $x = -0.5$, $x = 3.3$, or $x = 5.3$

c) $x = -0.6$ or $x = 2.0$

d) $x = -0.4$ or $x = 1.6$

e) $x = -0.1$, $x = 3.2$ or $x = 4.9$

f) $[1.3, 1.3] \cup (-1, -0.7]$

g) 0.5

h) all reals except -0.5, 3.3 and 5.3

i) $(-2, -0.4] \cup [3.3, 5.3]$

11. a) -0.5

b) -5

c) $\frac{1}{2}$

d) 11

13. domain: $(-\infty, \infty)$
one possibility: $f(x) = x+4$, $g(x) = x^2$

$g \circ f = (x+4)^2$

15. domain: $[0,25) \cup (25,\infty)$
one possibility: $f(x) = \frac{4}{x-5}$, $g(x) = \sqrt{x}$
$g \circ f = \frac{2}{\sqrt{x-5}}$;

17. domain: all reals except 0
one possibility: $f(x) = x-8$, $g(x) = \frac{1}{x}$
$g \circ f = \frac{1}{x-8}$;

19. domain: $(-\infty,\infty)$
one possibility: $f(x) = 5 - \frac{1}{2}x$, $g(x) = x^3$
$g \circ f = \left(5 - \frac{1}{2}x\right)^3$;

21. domain: $(-\infty,\infty)$
one possibility: $f(x) = 6+x$, $g(x) = 2^x$
$g \circ f = 2^{6+x}$

23. domain: all reals except 0
one possibility: $f(x) = \frac{3}{x}$, $g(x) = 5x^2$
$g \circ f = \frac{45}{x^2}$

25. domain: all reals except $-\frac{9}{2}$
one possibility: $f(x) = \frac{3}{x}$, $g(x) = 2x+9$
$g \circ f = \frac{6}{x} + 9$;

27. domain: $(-\infty,-3] \cup [3,\infty)$
one possibility: $f(x) = \sqrt{x}$, $g(x) = x^2 - 9$
$g \circ f = x - 9, x \geq 0$;

29. $\sqrt{\frac{3x}{6x-1}}$

31. $\frac{3x^2-7}{x^2-1}$

33. $f(x) = x^2 + 1$

35. a) $f(x) = 36\pi + 2\pi x$
b) $g(x) = \sqrt{\frac{x}{\pi}}$
c) $h(x) = \sqrt{36+2x}$ which is the length of the radius as a function of the time since the circle started shrinking.

Section 2.2 (page 47)

1. $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$

3. $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -2$

5. domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$; $\lim_{x\to-\infty} g(x) = -\infty$, $\lim_{x\to\infty} g(x) = \infty$

7. domain: $[0,\infty)$; range: $[-5,\infty)$; $\lim_{x\to\infty} Q(x) = \infty$, $\lim_{x\to 0^+} Q(x) = -5$

9. domain: $[5,\infty)$; range: $[0,\infty)$; $\lim_{x\to\infty} k(x) = \infty$, $\lim_{x\to 5^+} k(x) = 0$

11. domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$; $\lim_{t\to-\infty} R(t) = -\infty$, $\lim_{t\to\infty} R(t) = \infty$

13. domain: $[-12,\infty)$; range: $(0,\infty)$; $\lim_{t\to\infty} g(t) = \infty$, $\lim_{t\to-12^+} g(t) = 0$

15. domain: all reals except 8; range: $(-\infty,\infty)$; $\lim_{x\to-\infty} j(x) = -\infty$, $\lim_{x\to\infty} j(x) = -\infty$

17. domain: all reals except 1 and 4; range: $(-\infty,\infty)$; $\lim_{r\to-\infty} p(r) = 0$, $\lim_{r\to\infty} p(r) = 0$

19. domain: $(-\infty,\infty)$; range: $[-1.842...,\infty)$; $\lim_{z\to-\infty} h(z) = \infty$, $\lim_{z\to\infty} h(z) = \infty$

21. domain: $(-\infty,-11] \cup [-2,\infty)$; range: $[0,\infty)$; $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = \infty$

23. domain: $(-\infty,\frac{1}{2}] \cup [1,\infty)$; range: $[0,\infty)$; $\lim_{t\to-\infty} f(t) = \infty$, $\lim_{t\to\infty} f(t) = \infty$

25. domain: all reals except -1; range: all reals except 0; $\lim_{x\to-\infty} j(x) = 0$, $\lim_{x\to\infty} j(x) = 0$

27. domain: $(-\infty,-\sqrt{5}] \cup [\sqrt{5},\infty)$; range: $[0,\mathrm{inf})$; $\lim_{z\to-\infty} g(z) = \infty$, $\lim_{z\to\infty} g(z) = \infty$

29. domain: $[-3,3]$; range: $[-3,0]$; $\lim_{x\to-3^+} f(x) = 0$, $\lim_{x\to 3^-} f(x) = 0$

31. domain: $[1-\sqrt{6},1+\sqrt{6}]$; range: $[0,\sqrt{6})$; $\lim_{t\to 1-\sqrt{6}^+} h(t) = 0$, $\lim_{t\to 1+\sqrt{6}^-} h(t) = 0$

33. 12

35. doesn't exist

37. 3

39. ∞

41. doesn't exist

43. $-\infty$

Section 2.3 (page 51)

1. increasing on $[-4,-2]$, decreasing on $[-2,2)$, local and global maximum of 5 at $x=-2$

3. increasing on $(-1,1]$ and $[4.3,5.5]$, decreasing on $[1,4.3]$, local and global maximum of 3 at $x=1$, local minimum of -1 at $x \approx 4.3$

5. increasing on $[-3,3]$ and $[11,12]$, decreasing on $(-6,-3]$ and $[3,11]$, local maximum of approximately 4.2 at $x=3$, a local minimum of approximately 3.1 at $x=-3$, and a local and absolute minimum of approximately -2.4 at $x=11$

9. at $x \approx 3.32$, f has a local maximum value of approximately -2.85

11. at $x=-2$, f has a local maximum value of 10 and at $x=1$, f has a local minimum value of $-\frac{7}{2}$

13. at $x=-1.6$, f has a local maximum value of approximately -1.381, at $x=0$, f has a local and absolute minimum value of -3, and at $x=-2$, f has an absolute minimum value of 0

Section 2.4 (page 54)

1. a) $(3, 7)$
b) $(3, -7)$

Section 2.5 (page 60)

1. power; domain: $(-\infty, \infty)$ range: $\{1\}$; even

3. transformation; domain: $(-\infty, \infty)$ range: $\{-\frac{5}{2}\}$; even

5. transformation; domain: $(-\infty, \infty)$ range: $(-\infty, \infty)$; neither

7. transformation; domain: $(-\infty, \infty)$ range: $[0, \infty)$; even

9. transformation; domain: $(-\infty, \infty)$ range: $(-\infty, 1]$; even

11. transformation; domain: $(-\infty, \infty)$ range: $(-\infty, \infty)$; odd

13. transformation; domain: $(-\infty, 0]$ range: $[0, \infty)$; neither

15. power; domain: $(-\infty, \infty)$ range: $(-\infty, \infty)$; neither

17. tranformation; domain: all reals except 0 range: all reals except -2; odd

19. transformation; domain: all reals except -2 range: $(0, \infty)$; neither

21. transformation; domain: $(-\infty, \infty)$ range: $[-3, \infty)$; even

23. $y = 3\sqrt{-x}$

25. $y = \frac{1}{x+2} - 6$

27. $y = -\sqrt{\frac{x}{5}}$

29. To check your graph in Desmos, you can use the syntax
$f(x) = \left\{x \leq -6 : -3, -6 < x < -3 : 3(x+5), x \geq -3 : x^{-1}\right\}$.
(This will allow you to check everything except whether your open and closed endpoints are correct.)

31. To check your graph in Desmos, you can use the syntax
$f(x) = \left\{x < -5 : 2(x+4) + 7, -5 < x < 1 : |x|, x > 1 : -x^{\frac{1}{2}}\right\}$.
(This will allow you to check everything except whether your open and closed endpoints are correct.)

33. $f(x) = \begin{cases} -\frac{7}{3}(x+3) - 4 & , \quad x \leq -3 \\ x^{1/3} & , \quad x > -3 \end{cases}$; domain: $(-\infty, \infty)$, range: $[-4, \infty)$

35. $f(x) = \begin{cases} \frac{4}{3}(x+4) + 5 & , \quad x < -4 \\ x^{1/2} & , \quad x \geq 0 \end{cases}$; domain: $(-\infty, -4) \cup [0, \infty)$, range: $(-\infty, \infty)$

37. To obtain g, the graph of f is (1) moved 5 to the left, (2) reflected over the x-axis, and (3) moved up 3

39. To obtain g, the graph of f is (1) reflected over the y-axis, (2) stretched vertically by a factor of 2, and (3) moved down 3

41. a) $(12, -10)$
b) $(-12, -5)$
c) $(9, -5)$
d) $(36, -5)$

43. a) $(6, 8)$
b) $(-3, 12)$
c) $(-\frac{3}{5}, 8)$
d) $(-6, -22)$

45. local min of -110 at $x = 54$

Section 2.6 (page 65)

1. a) $-\frac{5}{3}$
b) $-\frac{5}{3}$
c) 0
d) 0
e) $-\frac{2}{7}$
f) $\frac{3}{2}$

7. $\frac{P(d)-P(c)}{d-c}$

9. 0

11. -4

13. $-\frac{1}{2}$

15. $-20a - 10h$

17. $4a + 2h - 7$

19. $6a + 3h - 2$

21. $-\frac{1}{a^2+ah}$

23. $\frac{a^2+ah-1}{a(a+h)}$

25. $-\frac{3}{10(2+c)}$

27. $-\frac{5}{12}$°F/min

29. $-\frac{3}{4(a-1)}$

31. avg rate of change is $a + b$

Section 2.7 (page 69)

3. $(7, -3)$

5. $\frac{x^3+5}{2}$

7. $\frac{2}{x} - \frac{2}{3}$

9. $\left(\frac{3}{x} - \frac{1}{2}\right)^{5/3}$

11. $x^2 - 2x - 3, x \leq 1$

13. $\frac{3x+5}{2x-1}$

15. $w(x) = x^3 - 367$

Section 2.8 (page 72)

1. a) domain: $(-4, 6]$ range: $[-3.2, 5.2]$

b) $[-1.7, 1.7]$, $[4.5, 6]$

c) f has a local and global minimum of approximately -3.2 at $x \approx -1.8$, a local and global maximum of approximately 5.2 at $x \approx 1.8$, and a local minimum of approximately 0.7 at $x \approx 4.4$

d) $x \approx -3.5$ or $x \approx -0.2$

e) $x \approx 0.8$, $x \approx 3.0$, $x \approx 5.6$

f) $(-4, 0] \cup [4, 5]$

g) 1

h) $[-3.5, -0.2]$

2. a) $Q : (-\infty, \infty); R : (-\infty, \frac{1}{2}]$

b) $\sqrt{2x^3 - 5}$

c) $\lim_{x\to-\infty} Q(x) = \infty$, $\lim_{x\to\infty} Q(x) = -\infty$, $\lim_{x\to-\infty} R(x) = \infty$, $\lim_{x\to\frac{1}{2}^-} Q(x) = 0$

3. a) even

b) even

c) odd

d) neither

4. a) $(-3, 16)$ and $(5, 12)$

b) $(6, -20)$ and $(-10, 8)$

c) $(6, 20)$ and $(-10, -8)$

5. $3\sqrt{-x} - 7$

8. a) transformation of x^{-1}; shift left 7, stretch vertically by a factor of 10, reflect over x-axis, move up 25

b) asymptotes are $x = -7$ and $y = 25$; $\lim_{x\to-\infty} P(x) = 25$, $\lim_{x\to\infty} P(x) = 25$, $\lim_{x\to-7^-} P(x) = \infty$, $\lim_{x\to-7^+} P(x) = -\infty$

c) $\frac{165-7x}{x-25}$

13. a) always

b) sometimes

Chapter 3

Section 3.1 (page 80)

1. $b > 1$

3. $b > 1$

5. $e^{3/5} = 9x$, $x = 0.202$

7. 2^{3x}

9. $2^{0.5x}$

11. $2^{(\log_2 \pi)x}$

13. domain: $(-\infty, \infty)$; range: $(0, \infty)$; asymptote: $y = 0$; intercept: (0,1); $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = \infty$; $g^{-1}(x) = \log_3 x$

15. domain: $(-\infty, \infty)$; range: $(0, \infty)$; asymptote: $y = 0$; intercept: (0,1); $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$; $g^{-1}(x) = -\log_3 x$

17. domain: $(-\infty, \infty)$; range: $(0, \infty)$; asymptote: $y = 0$; intercept: (0,2); 0 $\lim_{x\to\infty} f(x) = \infty$; $g^{-1}(x) = \log_3(\frac{x}{2})$

19. domain: $(0, \infty)$; range: $(-\infty, \infty)$; asymptote: $x = 0$; intercept: (1,0); 0 $\lim_{x\to\infty} f(x) = \infty$; $g^{-1}(x) = 2^x$

21. domain: $(-\infty, 0)$; range: $(-\infty, \infty)$; asymptote: $x = 0$; intercept: $(-1, 0)$; $\lim_{x\to-\infty} f(x) = \infty$; $g^{-1}(x) = -2^x$

23. domain: $(0, \infty)$; range: $(-\infty, \infty)$; asymptote: $x = 0$; intercept: (1,0); $\lim_{x\to\infty} f(x) = -\infty$; $g^{-1}(x) = \left(\frac{1}{2}\right)^x$

25. domain: $(-\infty, \infty)$; range: $(-\infty, 8)$; asymptote: $y = 8$; intercepts: (0,7) and $(\log_3 8, 0)$; $\lim_{x\to-\infty} f(x) = 8$, $\lim_{x\to\infty} f(x) = -\infty$; $g^{-1}(x) = \log_3(8 - x)$

27. domain: $(-3, \infty)$; range: $(-\infty, \infty)$; asymptote: $x = -3$; intercepts: $(-2, 0)$ and $(0, \log_4 3)$; $\lim_{x\to\infty} f(x) = \infty$; $g^{-1}(x) = 4^x - 3$

29. domain: $(5, \infty)$; range: $(-\infty, \infty)$; asymptote: $x = 5$; intercept: $\left(5\frac{1}{16}, 0\right)$; $\lim_{x\to\infty} f(x) = \infty$; $g^{-1}(x) = 5 + 4^{x-2}$

31. Graph your equation with the help of technology to see if you've got it.

33. Graph your equation with the help of technology to see if you've got it.

35. Graph your equation with the help of technology to see if you've got it.

37. Graph your equation with the help of technology to see if you've got it.

39. $f(x) = 3 \cdot 2^x$

Section 3.2 (page 93)

3. a) \$16,576.58

b) \$8629.19

5. a) \$55.78

b) \$52.57

7. a) $\$75,181.51$

b) $\$75,861.11$

c) $\$76,018.48$

d) $\$76,098.08$

9. a) 5.9274%

b) 5.9567%

c) 5.9710%

d) 5.9715%

11. $a < c < b < d$

19. a) $P(t) = 76 \cdot 2^{t/50}$; $349, 204, 300$

b) Our model suggests the population would reach 500 million approximately 136 years after 1900. Thus, it predicts the population will reach 500 million in 2036.

21. a) $P(t) = 151,325,798\left(\frac{76,212,168}{151,325,798}\right)^{t/50}$; $33,462,465$

b) 2080

23. a) 1,015,000; 1,030,225; 1,045,678

b) $P(t) = 1,000,000(1.015)^t$

c) 1,450,945

25. a) $A(t) = 500(0.2)^{t/26}$

b) 34.9 g

c) 72.7 years

d) 11.2 years

27. $A(x) = 300\left(\frac{1}{2}\right)^{x/16.50}$

29. a) $f(t) = 21.00\left(\frac{1}{2}\right)^t$

b) 118.79g

c) 2.21g

d) 5:24 p.m.

31. 729.30 mins

33. a) It takes 14 hours for the population to triple

b) 8767

c) $P(t) = 12000e^{0.07847t}$

d) 557.54%

35. a) It takes 9 hours for the population to increase by a factor of 5.

b) 458

c) 12:31 pm

d) 19.58

e) 7210.04

f) $P_o = 1600$,$a \approx 3.876$

Section 3.3 (page 99)

1. a) $f(x) = -4x(x-3) - 10$

c) $f(x) = -4\left(x - \frac{3+i}{2}\right)\left(x - \frac{3-i}{2}\right)$

3. $Q(x) = 2\left(x + \frac{3}{2}\right)(x+1)$; $Q(x) = 2\left(x + \frac{5}{4}\right)^2 - \frac{1}{8}$

5. $Q(x) = 5\left(x + \frac{2}{5}\right)(x+1)$; $Q(x) = 5\left(x + \frac{7}{10}\right)^2 - \frac{9}{20}$

7. $Q(x) = \left(x - \frac{1}{2} + i\frac{\sqrt{15}}{2}\right)\left(x - \frac{1}{2} - i\frac{\sqrt{15}}{2}\right)$; $Q(x) = (x - \frac{1}{2})^2 + \frac{15}{4}$

9. intercepts: $(0,14)$, $(-3-\sqrt{2},0)$, $(-3+\sqrt{2},0)$; domain: $(-\infty,\infty)$; range: $[-4,\infty)$

11. intercepts: $(0,15)$, $(3,0)$, $(-5,0)$; domain: $(-\infty,\infty)$; range: $(-\infty,16]$

13. intercepts: $(0,48)$, $(6,0)$, $(-4,0)$; domain: $(-\infty,\infty)$; range: $(-\infty,50]$

15. intercepts: $(0,8)$, $(2,0)$; domain: $(-\infty,\infty)$; range: $[0,\infty)$

17. intercepts: $(0,13)$, $(2+\sqrt{17},0)$, $(2-\sqrt{17},0)$; domain: $(-\infty,\infty)$; range: $(-\infty,17]$

19. no x-intercepts, y-intercept: $(0,3)$; domain: $(-\infty,\infty)$; range: $[3\frac{7}{8},\infty)$

21. $f(x) = -\frac{1}{4}(x+2)(x-12)$

23. $f(x) = -\frac{2}{3}(x-3)(x-6)$

25. Graph your equation with the help of technology to see if you've got it.

27. Graph your equation with the help of technology to see if you've got it.

29. 1 sec, 28 ft

31. 1.5 sec, 47 ft

33. 33

35. 1

Section 3.4 (page 104)

3. Graph your equation with the help of technology to see if you've got it.

5. Graph your equation with the help of technology to see if you've got it.

7. Graph your equation with the help of technology to see if you've got it.

9. $(-\infty,-1] \cup [2,2] \cup [3,\infty)$

13. $\left(-\infty,-\frac{1}{2}\right) \cup \left(-\frac{1}{2},0\right) \cup (2,6)$

15. $(-\infty,4] \cup \left[\frac{3}{7},1\right] \cup [2,\infty)$

17. $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$

19. $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to\infty} g(x) = -\infty$

21. The zeros are -2, $\pm 2i\sqrt{2}$

25. 326.60 in^3

27. b) domain: $(-\infty,\infty)$, range: $[-639.98,\infty)$

29. b) domain: $(-\infty,\infty)$, range: $(-\infty,\infty)$

Section 3.5 (page 108)

1. $\lim_{x\to-\infty} f(x) = 2$, $\lim_{x\to\infty} f(x) = 2$
$\lim_{x\to-3^-} f(x) = \infty$, $\lim_{x\to-3^+} f(x) = -\infty$
$\lim_{x\to1^-} f(x) = -\infty$, $\lim_{x\to1^+} f(x) = \infty$

7. One possibility is $f(x) = \frac{(x-7)(x+2)}{(x-1)(x+2)}$. Graph your result to check if it is another possibility.

9. One possibility is $f(x) = \frac{3x-5}{x+1}$. Graph your result to check if it is another possibility.

11. One possibility is $f(x) = \frac{x}{x^2+1}$. Graph your result to check if it is another possibility.

13. intercepts: $(-\frac{3}{2}, 0)$, $(0, \frac{3}{8})$, asymptotes: $y = 0$, $x = -2$, $x = -4$. There are no holes because the numerator and denominator have no common factors.

15. intercepts: $(2, 0)$, $(-2, 0)$, $(0, -4)$, asymptotes: $y = \frac{1}{9}$. No v.a. b/c denominator is never 0. No holes because the numerator and denominator have no common factors.

17. intercepts: $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $(0, -\frac{1}{5})$, asymptotes: $y = 4$, $x = -\sqrt{5}$, $x = \sqrt{5}$. No holes because there are no common factors of numerator and denominator.

19. Did you graph the hole? Domain is $\{x \mid x \neq 5\}$. Range is $\{y \mid y \neq 4\}$

21. Did you graph the hole? Domain is all reals except -4 and 4. Range is all reals except 1 and $\frac{5}{8}$

23. $y = 3x - 1$

Section 3.6 (page 112)

16. $[-6, -2] \cup \left[\frac{1}{3}, \frac{1}{3}\right]$.

17. $[-6, -2) \cup \left[\frac{1}{3}, \frac{1}{3}\right]$. Because $(x + 2)$ is now in the denominator, the zero associated with that factor will no longer make the overall fraction equal zero.

Chapter 4

Section 4.1 (page 119)

Section 4.2 (page 126)

3. $\theta \approx 1.25 + 2k\pi$ or $\theta \approx -1.25 + 2k\pi$

5. $\theta \approx 2.1 + 2k\pi$ or $\theta \approx 3.8 + 2k\pi$

7. $\theta \approx 0.9 + 2k\pi$ or $\theta \approx 2.25 + 2k\pi$

9. $\theta \approx -0.8 + 2k\pi$ or $\theta \approx 3.9 + 2k\pi$

11. $\sin(-\frac{3\pi}{7}$, $\sin(\frac{11\pi}{10}$, $\sin 3$, $\sin 2$, $\sin(-\frac{3\pi}{2}$

13. $\cos\frac{6\pi}{7}$, $\cos 2$, $\cos\frac{3\pi}{2}$, $\cos(-1)$, $\cos 0.1$

15. 0

17. 0

19. 0

21. 0

23. 0

25. $\frac{1}{2}$

27. $\frac{\sqrt{3}}{2}$

29. $\frac{\sqrt{2}}{2}$

31. $-\frac{\sqrt{2}}{2}$

33. $\frac{1}{2}$

35. $-\frac{\sqrt{2}}{2}$

37. $-\frac{1}{2}$

39. $-\frac{\sqrt{3}}{2}$

41. -1

43. $-\frac{1}{2}$

45. $\frac{\sqrt{2}}{2}$

47. $\frac{1}{2}$

49. $-\frac{\sqrt{2}}{2}$

51. 0

53. 0

55. 0

57. $-\frac{\sqrt{3}}{2}$

59. $\frac{\sqrt{3}}{2}$

61. -1

63. $-\frac{1}{2}$

65. $\frac{1}{2}$

67. 0

69. $-\frac{\sqrt{3}}{2}$

71. $-\frac{\sqrt{3}}{2}$

73. $\frac{1}{2}$

75. $-\frac{\sqrt{2}}{2}$

77. $x = \frac{3\pi}{4} + 2k\pi$ or $\frac{5\pi}{4} + 2k\pi$

79. $x = \pm\frac{\pi}{3} + 2k\pi$

81. $x = \frac{\pi}{3} + \frac{2k\pi}{3}$

83. $x = \frac{\pi}{4} + k\pi$

85. $x = \frac{\pi}{2} + k\pi$

87. $x = \frac{\pi}{6} + 2k\pi$ or $-\frac{\pi}{6} + 2k\pi$ or $k\pi$

89. $x = \frac{\pi}{2} + \frac{2k\pi}{3}$

91. $x = \frac{\pi}{3} + \frac{2\pi}{3} + k\pi$

93. no solution

95. $x = \frac{\pi}{10} + \frac{k\pi}{5}$

97. $x = \frac{7\pi}{6} + 2k\pi$ or $\frac{11\pi}{6} + 2k\pi$

99. $\cos(\pi - t) = -\frac{4}{5}$; $\cos(t + \frac{\pi}{2}) = -\frac{3}{5}$

105. amplitude is 6, period is 8

107. amplitude is 30, period is 20

125. 0

129. a) $[-3, 9]$

b) $\frac{3\pi}{20} + \frac{2k\pi}{5}$

c) 20

d) $-\frac{3\pi}{20}$, $\frac{\pi}{20}$, $\frac{5\pi}{20}$, $\frac{9\pi}{20}$

e) $\frac{\pi}{20} + 2k\pi$

131. $h(t) = 75 - 60\cos\left(\frac{\pi}{15}t\right)$

135. -2.802, -0.340

137. 3.785, 5.640

139. 0.955, 2.186, 4.097, 5.328

141. $0.124 + \frac{k\pi}{8}$, $0.269 + \frac{k\pi}{8}$

Section 4.3 (page 134)

1. 0

3. 1

5. undefined

7. $-\sqrt{2}$

9. 1

11. $-\sqrt{2}$

13. undefined

15. undefined

17. $-\frac{\sqrt{3}}{3}$

19. 0

21. 0

23. -1

25. -1

27. 0

29. $-\frac{2\sqrt{3}}{3}$

31. undefined

33. $-\sqrt{3}$

35. $\sqrt{3}$

37. $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\tan\theta = -\frac{\sqrt{2}}{4}$, $\cot\theta = -2\sqrt{2}$, $\sec\theta = -\frac{3\sqrt{2}}{4}$, $\csc\theta = 3$

41. $\sin\theta = -\frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$, $\tan\theta = -2$, $\sec\theta = \sqrt{5}$, $\csc\theta = -\frac{\sqrt{5}}{2}$

43. $\sin\theta = \frac{1}{\sqrt{3}}$, $\cos\theta = -\frac{2}{\sqrt{3}}$, $\tan\theta = -\frac{1}{\sqrt{2}}$, $\sec\theta = -\frac{\sqrt{3}}{2}$, $\csc\theta = \sqrt{3}$

47. approximately $196.3°$and $-163.7°$are two possible solutions. Add or subtract multiples of $360°$to find others.

57. $x = \frac{\pi}{3} + \frac{2k\pi}{3}$

Section 4.4 (page 141)

1. $\sin y - 1$

3. 3

23. $-\frac{\sqrt{3}}{2}$

25. -1

27. $-\frac{\sqrt{2}}{2}$

29. $\frac{1}{2}$

31. a) $\frac{\sqrt{6}-\sqrt{2}}{4}$

b) $\frac{\sqrt{6}-\sqrt{2}}{4}$

35. $\sin 2\theta = -\frac{24}{25}$; $\cos 2\theta = \frac{7}{25}$; $\tan 2\theta = -\frac{24}{7}$

37. 4 or $-\frac{3}{2}$

39. $\frac{\pi}{6}$, $\frac{5\pi}{6}$

41. $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$

Section 4.5 (page 144)

5. $\frac{\pi}{2}$

7. $\frac{\pi}{4}$

9. $\frac{5\pi}{6}$

11. π

13. $\frac{3\pi}{4}$

15. undefined

17. $\frac{\pi}{3}$

19. undefined

21. $\frac{9}{41}$

23. $\frac{65}{56}$

25. $\frac{2}{\sqrt{3}}$

27. $\frac{1}{4}$

29. $\frac{2}{\sqrt{3}}$

31. $\frac{63}{65}$

Section 4.6 (page 147)

15. $x = \frac{\pi}{20} + \frac{k\pi}{10}$

16. $x = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$ or $\frac{\pi}{2} + k\pi$

17. a) $x = \frac{\pi}{2} + \frac{2k\pi}{3}$

b) $x = \frac{\pi}{20} + \frac{k\pi}{10}$

Chapter 5

Section 5.1 (page 149)

3. $96.38°$

5. 6.5 m

13. 134.6 mi

Section 5.2 (page 155)

11. $\left(2, -\frac{\pi}{2}\right)$; $(0, -2)$

13. $\left(2, \frac{\pi}{4}\right)$; $\left(\sqrt{2}, \sqrt{2}\right)$

15. $(5.83, 120.96°)$

17. $(4.12, -14.04°)$

19. 9

21. $2\sqrt{15}$

23. $\sqrt{3}$

25. a) $e^{i\pi/2}$

b) $e^{-\pi/2}$

Chapter 6

Section 6.1 (page 166)

1. $|\overrightarrow{r}| = 10$, $\overrightarrow{r} = \langle -8, -6 \rangle$

3. b) -7
c) $\sqrt{74}$
d) $\sqrt{53}$
e) $A(-6, 10)$, $C(-8, 1)$
f) $(1, 1)$

5. a) $\langle -4, 1, -3 \rangle$
b) $\langle 4, -1, 3 \rangle$
c) $\sqrt{26}$
d) $\sqrt{26}$
e) $\langle -\frac{2}{13}, \frac{1}{26}, -\frac{3}{26} \rangle$

7. a) $\langle 6, -2, -5 \rangle$
b) $\langle -6, 2, 5 \rangle$
c) $\sqrt{65}$
d) $\sqrt{65}$
e) $\langle \frac{6}{\sqrt{65}}, -\frac{2}{\sqrt{65}}, -\frac{\sqrt{65}}{13} \rangle$

9. a) $\overrightarrow{OC} = \begin{pmatrix} 10 \\ -4 \end{pmatrix} + t \begin{pmatrix} 24 \\ 36 \end{pmatrix}$
b) 16 miles east and 5 miles north
c) 6 miles east and 10 miles south
e) $\frac{4}{21}$ of an hour before noon (approximately 11:48:34 am) she is approximately 10.54 miles from home

11. $\overrightarrow{r} = \begin{pmatrix} 0 \\ 3 \\ -10 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$; $\begin{matrix} x(t) = -3t \\ y(t) = 3 + 4t \\ z(t) = -10 - t \end{matrix}$

13. $a = 4$, $b = -2$

15. $\langle \frac{40}{41}, \frac{-9}{41} \rangle$

17. $a = \frac{\sqrt{23}}{6}$ or $-\frac{\sqrt{23}}{6}$

19. $\langle 140, -140, 70 \rangle$

21. a) $\langle 2, 5, -1 \rangle$
b) $\sqrt{30}$
c) $\langle \frac{3}{7}, -\frac{6}{7}, \frac{2}{7} \rangle$
d) $\langle -\frac{30}{7}, \frac{60}{7}, -\frac{20}{7} \rangle$

25. $\frac{4}{5}$

27. a) $\begin{matrix} x(t) = -10 + 6t \\ y(t) = 30 - 8t \end{matrix}$
b) 10 km/hr
c) $10\sqrt{10}$ km
d) $\langle -22, 14 \rangle$
e) 3:40 p.m.
f) $\overrightarrow{s} = \langle 2, 5 \rangle + t1, -1$
g) $\overrightarrow{v} = \langle -12, 25 \rangle + t5, -7$
h) $\langle 0, 0, -0.3 \rangle$

Section 6.2 (page 171)

1. a) 3
b) 4
c) $83.6°$

Section 6.3 (page 174)

Chapter 7

Section 7.1 (page 175)

1. 1

3. 2018

5. 10,100

7. 120

13. 165

17. 39,916,800

19. a) 22,100
b) 4804
c) 64
d) 286

21. 455

23. a) 1326
b) 676

27. WolframAlpha can do this one.

Section 7.2 (page 177)

17. 0.510

Chapter 8

Section 8.1 (page 183)

Section 8.2 (page 184)

Chapter 9

Section 9.1 (page 187)

1. $-\infty$

3. 0

5. 1

7. ∞

9. $\frac{1}{4}$

11. $2x+1$

13. $2+\frac{\pi}{6}$

15. 0

17. DNE

Section 9.2 (page 188)

Chapter 10

Section 10.1 (page 191)

1. One is $y+29=4(x+10)$. To find another, think about how much y will change by if x changes by 1.

3. $-\frac{1}{1000}$

5. 9

7. 32

9. -4

11. undefined

13. 3

15. $\frac{5}{2}$

17. $\frac{2}{3}$

19. $-\frac{2}{3}$

21. $-1-i$

25. x-int: $\log_3 90$, which is between 4 and 5, y-int: -89

27. $\dfrac{-x}{3(2x+3)}$

29. $\frac{195}{1-2i}$

31. $\lim\limits_{x\to-\infty} f(x)=\infty$, $\lim\limits_{x\to\infty} f(x)=4$

33. $(-\infty,4]$

35. $[2,5)\cup(5,\infty)$

39. $f(x)=\begin{cases} 7+\frac{5}{3}(x+4) & , \quad x<-4 \\ (x+4)^{1/2}-2 & , \quad x\geq -4 \end{cases}$

41. a) One possibility is $P(x)=\dfrac{1}{x}$, $Q(x)=x^2$, $R(x)=x-6$

b) $\lim\limits_{x\to-\infty} f(x)=0$, $\lim\limits_{x\to\infty} f(x)=0$

c) $x=6$, $y=0$

45. a) One possibility is $P(x)=x-3$, $Q(x)=\dfrac{1}{x}$, $R(x)=x+2$

b) $\lim\limits_{x\to-\infty} g(x)=-3$, $\lim\limits_{x\to\infty} g(x)=-3$

c) $x=-2$, $y=-3$

47. a) $(-1,-12)$ and $(4,-10)$

b) $(\frac{1}{4},-5)$ and $(-1,6)$

c) $(-1,12)$ and $(4,-10)$

d) $(-1,12)$ and $(4,10)$

49. a) local maximum of 28 at $x=18$

b) 7

c) $(-\infty,16)\cup(20,\infty)$

53. one example is $f(x)=x^2$

55. $a>1$

57. a) domain: $(-\infty,\infty)$, range: $(-\infty,5)$, y-int: 4, y-int: $\log_3 5$, asymp: $y=5$

b) $g^{-1}(x)=\log_3(5-x)$

59. b) domain: $(0,\infty)$, range: $(-\infty,\infty)$, asymptote: $x=0$

c) $h(x)=\left(\frac{1}{2}\right)^x$

d) $[-1,0]$

61. a) 5.904%

b) 5.919%

63. a) \$4.49

b) \$3.52

65. $f(x)=\frac{1}{20}(x+4)(x+10)$, range: $\left[-\frac{9}{2},-\infty\right)$

67. a) $g(x)=-2\left(x-\frac{5}{4}\right)^2+\frac{49}{8}$

b) $(0,\frac{5}{2})$

c) $5-4a-2h$

71. $-\frac{2\sqrt{3}}{3}$

79. $\sqrt{15}$

83. amp: 10, period: 10, $f(x)=-6-10\cos\left(\frac{\pi}{5}x\right)$

85. domain: all reals except $k\pi$, range: $(-\infty,\infty)$, period: π

93. 19